Dynamics of particle deposition on a disordered substrate. II. Far-from-equilibrium behavior

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The deposition dynamics of particles (or the growth of a rigid crystal) on a disordered substrate at a finite deposition rate is explored. We begin with an equation of motion which includes, in addition to the disorder, the periodic potential due to the discrete size of the particles (or to the lattice structure of the crystal) as well as the term introduced by Kardar, Parisi, and Zhang (KPZ) to account for the lateral growth at a finite growth rate [Phys. Rev. Lett. **56**, 889 (1986)]. A generating functional for the correlation and response functions of this process is derived using the approach of Martin, Siggia, and Rose [Phys. Rev. A **8**, 423 (1973)]. A consistent renormalized perturbation expansion to first order in the non-Gaussian couplings requires the calculation of diagrams up to three loops. To this order we show, for this class of models which violates the the fluctuation-dissipation theorem, that the theory is renormalizable. We find that the effects of the periodic potential and the disorder decay on very large scales and asymptotically the KPZ term dominates the behavior. However, strong nontrivial crossover effects are found for large intermediate scales.

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I. INTRODUCTION

A. General

A few years ago the near equilibrium dynamics of a growing crystalline surface was elucidated [1, 2]. It was found that a roughening transition occurs at $T = T_r$ between a high-temperature rough phase and a phase with a flat surface for $T < T_r$. The mobility of the growing surface drops from a finite value to zero as $T \to T_r^+$. In the low-temperature phase the growth is "activated" with the formation of higher "islands" on top of the flat surface. A similar behavior occurs in the deposition of cubic particles with diffusion on a flat substrate. The transition in this case is as function of the inherent noise due to spatial and temporal fluctuations in the deposition.

In view of the existence of a low temperature (or low noise) flat phase, the question of how disorder in the substrate would change the behavior had to be addressed. We have initiated a comprehensive study of the related questions. A short letter which announced the surprising results was published elsewhere [3]. In a previous full paper [4] (denoted by I in the following) we have presented the detailed calculation and analysis for the near-equilibrium dynamics. In this regime the averaged growth rate is small. The equation of motion is derived from the Hamiltonian of the system. Both detailed-balance and the fluctuation-dissipation theorem (FDT) [5] hold. Yet we have found very nontrivial results: super-rough correlations, temperature-dependent dynamics exponent, and a nonlinear relation between the average growth rate and the driving force were found below a super-roughening transition temperature T_{sr}.

In the present paper, the second in the series, we present our detailed calculations and results for the dynamics far from equilibrium. In this case the equation of motion cannot be derived from a Hamiltonian. Detailed-balance and the FDT are both violated. That situation represents a much more serious theoretical challenge since even the very renormalizability of the process is questionable.

The equation of motion we analyze describes the deposition of cubic particles on a random substrate. It will also apply to the surface of a crystal if the solid is very rigid. Since the effects of the disorder in the substrate will be felt only up to a height h^* (which is larger if the solid is more rigid), our theory will apply as long as $h < h^*$.

The width of a growing surface w follows generally the scaling form [6, 7]

$$w(L,t) \sim L^{\alpha} f(t/L^{z}), \tag{1}$$

where t is the time and L is the linear size of the system. α is the roughening exponent and z is the dynamic exponent.

The leading difference between near- and far-from-equilibrium dynamics is due to the lateral growth. If the growth rate is finite, the lateral growth adds a term proportional to $(\vec{\nabla} h)^2$ to the equation of motion. This term was derived by Kardar, Parisi, and Zhang (KPZ) [8] as the most relevant term in the expansion in term of $\vec{\nabla} h$.

As the renormalization group (RG) analysis shows, the KPZ nonlinearity is marginally relevant in 2+1 dimensions. The asymptotic behavior is controlled by this nonlinearity, which violates the FDT. Consequently, the

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roughness of the growing surface is also affected by the nonlinearity. Several simulations showed $\alpha \sim 0.4$ [9,10] which is larger than that of the near-equilibrium case ($\alpha = 0$). Many variants of KPZ-related models have been studied in recent years [6,7,11–22].

The effects of such a term on the growing crystalline surface on a flat substrate was considered by Hwa, Kardar, and Paczuski (HKP) [23]. We review and revise their results in Sec. I B. Section II will be devoted to the derivation of the generating functional, and the description of the RG procedure. Sections III, IV, and V are devoted to the outline of the calculations of the different renormalization factors. Extensive details of these calculation are relegated to the appendixes. In Sec. VI the recursion relations are derived. Section VII is devoted to the analysis of the asymptotic and crossover behaviors which follows from these recursion relations. Our main conclusions are summarized in Sec. VIII.

B. Review of previous works

The system we shall study in this paper has three important and nontrivial ingredients: (i) the periodic potential, (ii) the disorder in the substrate, and (iii) the KPZ nonlinearity arising from the lateral growth in the presence of a finite driving force (or deposition rate). Paper I discussed the case in which (i) and (ii) are present. The major physical consequences, namely the existence of a super-rough "glassy phase" for $T < T_{sr}$ with intriguing static and dynamic properties, were discussed there and will not be elaborated further here. Our goal here is to see how the addition of (iii) (i.e., the KPZ term) modifies the behavior.

However, we can also take another point of view and ask how the addition of (ii) (i.e., the disorder in the substrate) is modifying the behavior found in the presence of (i) and (iii). The model which discussed the nonequilibrium growth on a flat surface in the presence of both the periodic potential and the KPZ nonlinearity was analyzed by HKP [23]. The equation of motion which describes the growing surface under these calculations is

$$\tilde{\mu}^{-1}\frac{\partial h(\vec{x},t)}{\partial t} = F + \nu[\nabla^2 h(\vec{x},t)] + \frac{\lambda}{2}(\vec{\nabla}h)^2 + \frac{\gamma y_2}{a^2}\sin[\gamma(h(\vec{x},t))] + \zeta(\vec{x},t). \quad (2)$$

In this equation $h(\vec{x},t)$ is the local height at time t; $\vec{x} = (x,y)$ are the coordinates in the $2d$ basal plane; μ is the microscopic mobility; F is the driving force; ν is the surface tension; λ is the coefficient of the KPZ nonlinearity (which is proportional to F); y_2 is the coefficient of the leading harmonies (higher harmonies are irrelevant); $\gamma = \frac{2\pi}{b}$, where b is the vertical lattice spacing; a is the horizontal lattice spacing; and $\eta(\vec{x},t)$ is the noise in the deposition (or due to thermal fluctuations) that obeys

$$\langle\zeta(\vec{x},t)\zeta(\vec{x}',t')\rangle = 2D\delta^2(\vec{x}-\vec{x}')\delta(t-t'). \quad (3)$$

HKP obtained a set of recursion relations from which they reached several conclusions. They found a critical temperature T_c. For $T > T_c$, y_2 decays slowly to zero and the large scale behavior is determined by the KPZ coupling. Approaching T_c from above, the linear-response macroscopic mobility (namely the ratio v/F where $v = \langle\frac{dh}{dt}\rangle$ is the averaged growth rate in the limit $F \to 0$) vanishes as $(\ln|T-T_c|)^{-\eta}$. In the low temperature phase $T < T_c$ they have found that y_2 grows indefinitely large and therefore have concluded that the surface is flat.

While reanalyzing their work we have discovered a term that was overlooked in their calculations and which might affect their latter conclusion. Indeed it may be shown that a term of the form $y_1\cos(2\pi h)$ is generated under renormalization from the contraction of the terms $\lambda(\vec{\nabla}h)^2$ and $y_2\sin(2\pi h)$. This term also feeds back into the renormalization of y_2.

The recursion relations to lowest order are

$$\frac{dy_1}{dl} = \left(2 - \frac{\pi\mu D}{\nu}\right)y_1 - \rho\frac{\lambda}{\nu}y_2, \quad (4)$$

$$\frac{dy_2}{dl} = \left(2 - \frac{\pi\mu D}{\nu}\right)y_2 + \rho\frac{\lambda}{\nu}y_1, \quad (5)$$

where $\rho = \frac{1}{2}[4\pi^2\ln(4/3)(\frac{\mu D}{\nu})^2 + (\frac{\mu D}{\nu})]$ (the lattice spacing is taken to unity for simplicity). The two harmonic terms may be combined into a single term:

$$|y|\sin[2\pi h + \vartheta(l)], \quad (6)$$

with $y^2 = y_1^2 + y_2^2$, and $\vartheta(l) = \tan^{-1} y_1/y_2$.

If we look at the flow of $|y|$ and ϑ we find that indeed $|y| \to \infty$ for $T < T_c$, which would indicate a flat surface if ϑ were to remain constant. However, the recursion relations imply that the phase shift angle is rotating with l like $\vartheta(l) = wl$, with an angular velocity $w \sim \lambda/\nu$. Since $l = \ln\tilde{b}$, where $\tilde{b}$ is the rescaling factor, it means an ever changing $\vartheta \sim w\ln\tilde{b}$, with rescaling.

Therefore it is not obvious that the surface is flat. Hopefully, higher-order terms in the recursion relations of y_1, y_2, and ν will help to identify with more confidence the nature of the low-temperature phase.

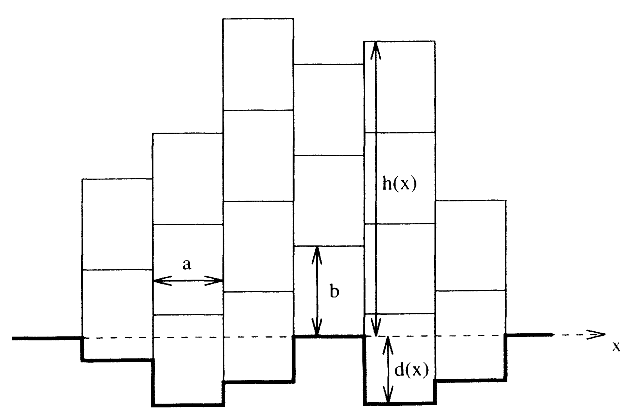

FIG. 1. A two-dimensional cut (along a lattice plane perpendicular to the disordered substrate) of the three-dimensional system.

C. Introducing the disorder

As we explained in I, and which is clear from Fig. 1, the disorder will shift the origin of the periodic potential by a random and uncorrelated amount at every point $\vec{x}$ in the basal plane [3, 4]. We take the deposited particle as having a rectangular shape with a square base of linear extent a and height b in the growth direction. Then the periodic potential becomes proportional to $\sin[\frac{2\pi}{b}(h(\vec{x},t)+d(\vec{x}))]$, where $d(\vec{x})$ are the local deviation in the height of the substrate. Let us denote the associate phase $\Theta(\vec{x}) = 2\pi d(\vec{x})/b$. It obeys

$$\langle e^{i\Theta(\vec{x})} e^{-i\Theta(\vec{y})}\rangle = a^2\delta^2(\vec{x}-\vec{y}). \qquad (7)$$

We have assumed that $d(\vec{x})$ are typically of order b (or larger) and that if correlations exist in $d(\vec{x})$ at different $\vec{x}$ they are at most short range (in which case they fall in the same universality class as the δ correlated disorder we study here). The equation of motion we need to study is therefore

$$\tilde{\mu}^{-1}\frac{\partial h(\vec{x},t)}{\partial t} = F + \nu[\nabla^2 h(\vec{x},t)] + \frac{\lambda}{2}(\vec{\nabla}h)^2 + \frac{\gamma y}{a^2}\sin[\gamma h(\vec{x},t)+\Theta(x)] + \zeta(\vec{x},t). \qquad (8)$$

II. GENERATING FUNCTIONAL AND BASIC DIAGRAMS

The Martin, Siggia, and Rose (MSR) [24–26] method is utilized to obtain the generating functional for the correlation and response theory. An auxiliary field $i\tilde{h}(\vec{x},t)$ is introduced to enforce the equation of motion through an integral representation of the δ function. Then the thermal noise and the disorder are averaged to yield the following generating functional:

$$\langle Z_\Theta[\tilde{J},J]\rangle_{disorder} = \int \mathcal{D}\tilde{h}\,\mathcal{D}h\,\exp\left\{\int d^2x dt\left[\tilde{D}\tilde{\mu}^2\tilde{h}^2 - \tilde{h}\left(\frac{\partial}{\partial t}h - \tilde{\mu}\nu\nabla^2 h - \tilde{\mu}\frac{\lambda}{2}(\vec{\nabla}h)^2\right)\right] + \frac{\tilde{\mu}^2\gamma^2\tilde{g}}{2a^2}\int\int d^2x\,dt\,dt'\,\tilde{h}(\vec{x},t)\tilde{h}(\vec{x},t')\cos[\gamma(h(\vec{x},t)-h(\vec{x},t'))]\right\}. \qquad (9)$$

A. Basic diagrams

From now on we change notation: h to ϕ and $\tilde{h}$ to $\tilde{\phi}$. As explained in detail in I the Gaussian (quadratic) part of the "action" gives rise to the bare response function

$$\langle\phi(\vec{q},\omega)\tilde{\phi}(-\vec{q},-\omega)\rangle = \frac{1}{\mu(q^2+m^2)+i\omega}, \qquad (10)$$

which is depicted in Fig. 2. The bare correlation function is :

$$\langle\phi(\vec{q},\omega)\phi(-\vec{q},-\omega)\rangle = \frac{2D\mu^2}{[\mu(q^2+m^2)]^2+\omega^2}, \qquad (11)$$

where m, the mass of the field ϕ, is introduced to control the infrared divergences in the Feynman integrals. (In the two-dimensional regularization, it is notorious [25] that their infrared and ultraviolet divergences will mingle together without introducing masses for the fields.)

In the momentum and time representation, they are given by [26]

$$\langle\phi(\vec{q},t)\tilde{\phi}(-\vec{q},t')\rangle = \theta(t-t')e^{-\mu(q^2+m^2)(t-t')}, \qquad (12)$$

$$\langle\phi(\vec{q},t)\phi(-\vec{q},t')\rangle = \frac{D\mu}{q^2+m^2}e^{-\mu(q^2+m^2)|t-t'|}, \qquad (13)$$

where $\theta(t) = 1$ for $t > 0$ and $\theta(t) = 0$ for $t < 0$. The basic vertex $\tilde{\phi}\tilde{\phi}'\cos(\phi-\phi')$ and $\tilde{\phi}(\vec{\nabla}\phi)^2$ are drawn in Figs. 3 and 4, respectively.

B. Renormalization group procedure

We follow the minimal subtraction scheme. The renormalization parameters are related to the bare ones by the so-called Z factors. The minimal scheme consists in extracting from the diagrams only the divergent parts which are all expressed in terms of functions of the dimensionality of the system.

The bare and renormalized vertex functions can be related by factors of Z_ϕ, $Z_{\tilde{\phi}}$. For instance,

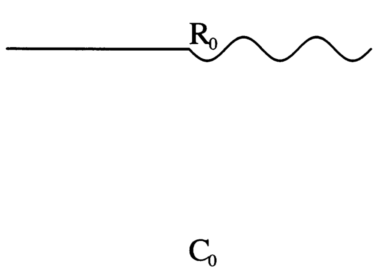

FIG. 2. The Feynman diagram for the correlation function and response function.

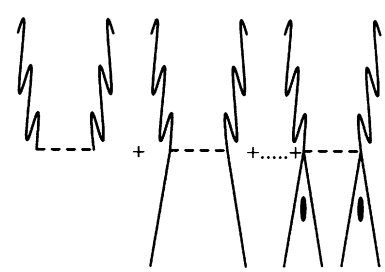

FIG. 3. The Feynman diagram representing $\tilde{\phi}(\vec{x},t)\tilde{\phi}(\vec{x},t')\cos[\phi(\vec{x},t)-\phi(\vec{x},t')]$.

$$\Gamma^R_{N,L}(q,\omega;\varsigma_R,m_R,\kappa) = (Z_{\tilde{\phi}})^{\frac{N}{2}}(Z_\phi)^{\frac{L}{2}} \times\Gamma_{N,L}(q,\omega;\varsigma_0,m_0,a), \quad (14)$$

where ς_R and ς_0 label renormalized parameters $(g,\mu,\ldots)$ and bare parameters $(g_0,\mu_0,\ldots)$, respectively. q and ω are the external momentum and frequency, respectively. In the corresponding vertex function, a is a short-distance cutoff and κ is a mass scale. Here $\Gamma_{N,L}$ stands for the vertex function with N external $\tilde{\phi}$ lines and L external ϕ lines. The factors, Z_ϕ and $Z_{\tilde{\phi}}$, are set to remove the divergent parts of the vertex function Γ.

The following renormalization constants are defined through the relations between the bare and the renormalized couplings [25, 27–29]:

$$D_0 = Z_D D, \quad g_0 = Z_g g, \quad \lambda_0 = Z_\lambda \lambda, \quad (15)$$

$$m_0^2\phi^2 = m^2\phi_R^2, \quad \gamma_0^2\phi^2 = \gamma^2\phi_R^2,$$
$$\phi^2 = Z_\phi\phi_R^2, \quad \tilde{\phi}^2 = (\tilde{Z}_{\tilde{\phi}})\tilde{\phi}_R^2, \quad (16)$$

where $\tilde{Z}_{\tilde{\phi}} = Z^2_{\tilde{\phi}}$.

In the next section we concentrate on the procedures to calculate the Z factors. The details are relegated to the appendixes.

The renormalized perturbation theory even to lower nontrivial orders in g and λ has to be consistent order by order in γ. It also requires the calculations of the Feynman diagrams [25, 29] with up to three loops.

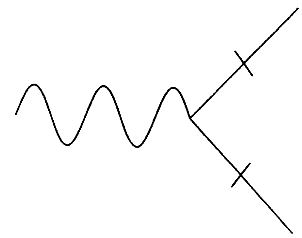

FIG. 4. The basic diagram for λ.

III. CALCULATIONS OF Z_D AND Z_μ

The renormalization of μ is not affected in the presence of the KPZ nonlinearity since the associated vertex function $\Gamma_{1,1}$ comes with one external $\tilde{\phi}$ and one external ϕ and the basic vertex λ contains derivatives on its two ϕ legs. Thus the factor Z_μ remains the same as in paper I (equilibrium dynamics), and so does the recursion relation for μ. On the other hand, the parameter D suffers additional renormalization of order λ^2. Basically, the renormalization of D from λ^2 is the same as that of D encountered in the KPZ model. Here, we still focus on the same vertex function $\Gamma_{2,0}$ as we did in the previous paper (I).

Obviously, the first nontrivial contribution begins from the second order in λ. As shown in Fig. 5, the vertex function is modified by the associated integral with λ^2. The corresponding integral is given by

$$\left(\frac{\mu_0\lambda_0}{2}\right)^2 \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d^2\vec{p}\int_{-\infty}^{\infty} d\Omega\frac{1}{2\pi}\frac{(2D_0\mu_0^2)^2p^2p^2}{[\mu_0^2(p^2+m^2)^2+\Omega^2][\mu_0^2(p^2+m^2)^2+\Omega^2]}$$
$$= \frac{1}{4}\lambda_0^2(\mu_0^3D_0^2)\int_{-\infty}^{\infty} d^2\vec{p}\frac{1}{(2\pi)^2}\frac{1}{(p^2+m^2)}$$
$$= \frac{1}{4}\lambda_0^2(D_0^2\mu_0^3)\left(-\frac{1}{4\pi}\ln(cm^2a^2)\right). \quad (17)$$

As illustrated in Fig. 5, the symmetry factor for this diagram is 2 and another factor 2 arises from the differentiation of the external legs. One more factor $\frac{1}{2}$ is due to the coefficient from the expansion of interaction with power 2 for the utilization of λ^2. With the combination of Eq. (D6) in paper I, we have

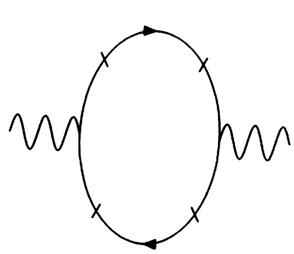

FIG. 5. The Feynman diagram for $\Gamma_{2,0}$ up to order λ^2.

$$-2D\mu^2 = (Z_{\tilde{\phi}})^2\left[-2D_0\mu_0^2 - \frac{D_0^2\mu_0^3}{2}\lambda^2\left(-\frac{1}{4\pi}\ln(cm^2a^2)\right) + \gamma^2\sqrt{c}g\mu_0\ln(cm^2a^2)\right]. \quad (18)$$

Therefore

$$Z_D = 1+\frac{D\mu\lambda^2}{4}\left(\frac{1}{4\pi}\ln(cm^2a^2)\right)+\frac{\gamma^2\sqrt{c g}}{2D\mu}\ln(cm^2a^2). \quad (19)$$

IV. THE CALCULATION OF Z_g

As in the harmonic model of paper I, we consider the vertex function $\Gamma_{2,0}(\vec{q},t;-\vec{q},t')$ in the limit $|t-t'|\to\infty$

[28]. The calculation of Z_g will be based on Eq. (8.3) of Goldschmidt and Schaub [28], where the renormalization factor is substituted in that equation to make $\Gamma_{2,0}(\vec{q},t;-\vec{q},t') \ (\vec{q}\to 0)$ finite. Therefore to find out Z_g is just to calculate the renormalization of $\Gamma_{2,0}$. The contributions to Z_g we need to sum are of order g^2 (as in paper I) and of order $\lambda^2 g$.

The combination of g and λ^2 leads to two types of diagrams, two-loop and three-loop diagrams. Some of the associated diagrams are canceled by each other as shown in Figs. 6 and 7. The other nonvanishing diagrams, including six two-loop and two three-loop diagrams, are shown in Figs. 8–15. The detailed calculations of two-loop integrals are given in Appendix A, where we also explain the cancellation of subdivergences of some diagrams. The sum of the leading and subleading divergences contributing to Z_g are listed in Eq. (22) (see the third and fourth terms). In Appendix B, we present the detailed calculation of the leading divergences in the three-loop diagrams and also show that they do not contain any subdivergence.

Now, what remains to complete the three-loop results is just to sum up the leading divergent terms in $\ln(cm^2a^2)$, which essentially contribute to the recursion relations. The $\ln(cm^2a^2)$ contribution of the diagram in Fig. 14 will be

$$\begin{aligned}-\tfrac{1}{2}&(I_{3.1-1}-2I_{3.1-2}-2I_{3.1-3})\\ &=\tfrac{3}{2}I_B+2I_D+4I_E+20\tfrac{1}{2}\ln(\tfrac{3}{4})-6[\ln(\tfrac{4}{3})]^2\\ &\quad-4\ln(2)-\tfrac{5}{2}(\ln 2)^2+22\ln(\tfrac{3}{2})\\ &\quad+3\Phi(1,2)-3\Phi(\tfrac{1}{2},2)-\Xi(\tfrac{1}{4},2)\ . \end{aligned} \tag{20}$$

The contribution from the diagram in Fig. 15 is:

$$\begin{aligned}I_{3.2-1}&+2I_{3.2-2}+2I_{3.2-3}\\ &=\frac{1}{-\frac{3\epsilon}{2}}[\tfrac{1}{4}I_C+I_D-2I_E+\tfrac{1}{2}\ln(\tfrac{3}{2})\\ &\quad-\tfrac{3}{4}\ln(\tfrac{4}{3})-\tfrac{1}{2}\ln(\tfrac{4}{3})]. \end{aligned} \tag{21}$$

Now we are in a position to calculate Z_g. Inserting the above calculation results in the self energy in Ref. [28], we obtain

$$\begin{aligned}Z_g&=1-\delta_0\ln(cm^2a^2)+\frac{\lambda^2\gamma^2(D\mu)^2}{8}\frac{[\ln(cm^2a^2)]^2}{16\pi^2}\\ &\quad-\frac{-5+11\ln(\frac{4}{3})}{16}\gamma^2(\mu D)^2\lambda^2\frac{\ln(cm^2a^2)}{(4\pi)^2}\\ &\quad+(-90.5)\gamma^4(D\mu)^3\lambda^2\frac{\ln(cm^2a^2)}{(4\pi)^3}. \end{aligned} \tag{22}$$

By using $\delta_0=\delta+\frac{1}{4}\lambda^2(D\mu)^2\gamma^2\frac{\ln(cm^2a^2)}{(4\pi)^2}$, we can calculate the recursion relation of g, as will be shown in the next section. As we will see, the term $\ln(cm^2a^2)^2$ is cancelled when we derive the Callan-Symanzik (CS) [25, 29] equation. Thus the scaling equation is consistent.

V. THE CALCULATION OF Z_λ

For the calculation of Z_λ, we consider the vertex function $\Gamma_{1,2}$. For the diagrams in Figs. 16 and 17, one can write the associated integrals as

$$nla=\int_{-\infty}^{\infty}d^2\vec{k}\int_{-\infty}^{\infty}d\Omega\frac{2D\mu^2\vec{p}\cdot(\frac{\vec{p}}{2}-\vec{k})}{\{\mu[(-\frac{\vec{p}}{2}-\vec{k})^2+m^2]-i\Omega\}\{\mu^2[(\frac{\vec{p}}{2}-\vec{k})^2+m^2]^2+\Omega^2\}}f(\Omega) \tag{23}$$

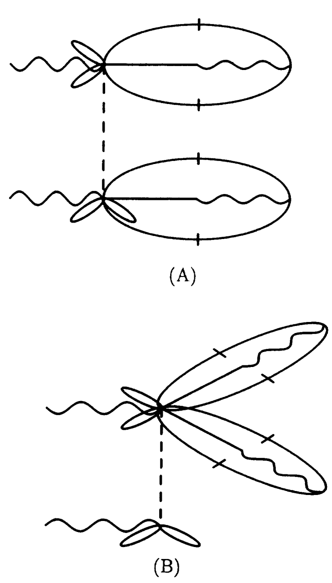

FIG. 6. Two mutually canceled two-loop Feynman diagrams.

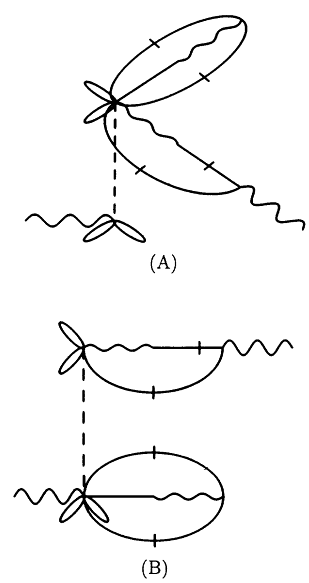

FIG. 7. Two mutually canceled two-loop Feynman diagrams.

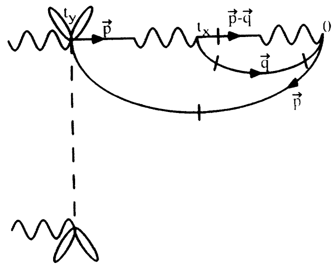

FIG. 8. A two-loop Feynman diagram: FD 2l1.

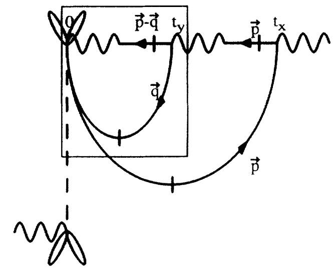

FIG. 12. A two-loop Feynman diagram: FD 2l5.

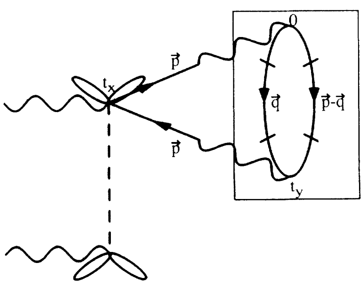

FIG. 9. A two-loop Feynman diagram: FD 2l2.

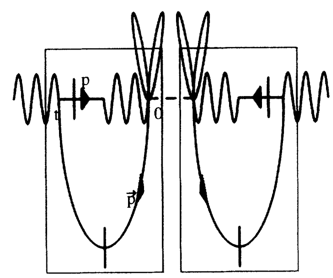

FIG. 13. A two-loop Feynman diagram: FD 2l6.

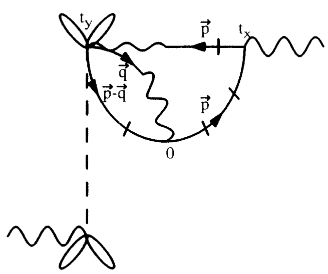

FIG. 10. A two-loop Feynman diagram: FD 2l3.

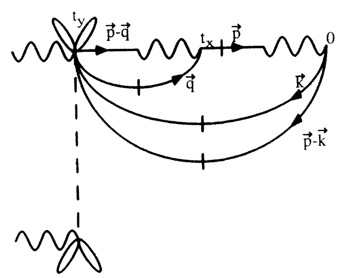

FIG. 14. A three-loop Feynman diagram: FD 3l1.

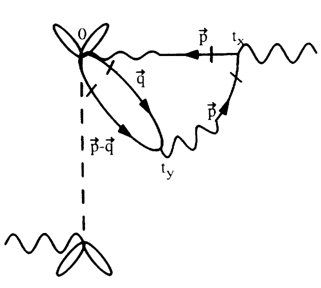

FIG. 11. A two-loop Feynman diagram: FD 2l4.

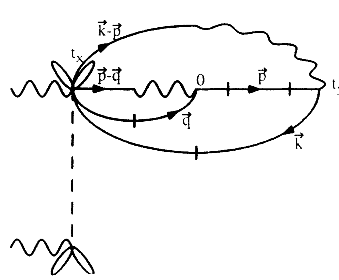

FIG. 15. A three-loop Feynman diagram: FD 3l2.

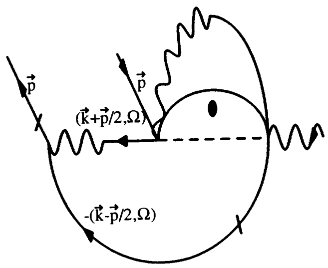

FIG. 16. A Feynman diagram: FD *nla*.

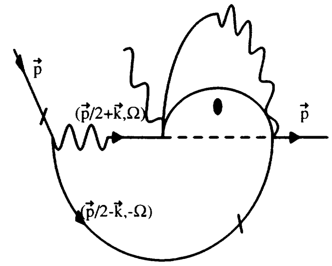

FIG. 17. A Feynman diagram: FD *nlb*.

and

$$nlb = \int_{-\infty}^{\infty} d^2\vec{k} \int_{-\infty}^{\infty} d\Omega \frac{2D\mu^2 \vec{p}\cdot(\frac{\vec{p}}{2}-\vec{k})}{\{\mu[(\frac{\vec{p}}{2}+\vec{k})^2+m^2]-i\Omega\}\{\mu^2[(\frac{\vec{p}}{2}-\vec{k})^2+m^2]^2+\Omega^2\}} f(\Omega), \tag{24}$$

where

$$f(\Omega) = \gamma^2 \int_{-\infty}^{\infty} dt e^{i\Omega t}[R_0(0,t)e^{\gamma^2 C_0(0,t)}] = -\int_{-\infty}^{\infty} dt e^{i\Omega t}\frac{1}{\mu^2 D}[R_0(0,t)\gamma^2]e^{\gamma^2 C_0(0,t)}$$
$$= \frac{1}{\mu^2 D}(i\Omega)\int_0^{\infty} dt e^{i\Omega t}(e^{\gamma^2 C_0(0,t)}-1). \tag{25}$$

For simplicity, let $x = (\frac{\vec{p}}{2}+\vec{k})^2$ and $y = (\frac{\vec{p}}{2}-\vec{k})^2$. The summation of *nla* in Eq. (23) and *nlb* in Eq. (24) is proportional to $\int_{-\infty}^{\infty} d\Omega \frac{2x}{(x^2+\Omega^2)(y^2+\Omega^2)} f(\Omega)$.

With the help of Eq. (25), the frequency dependent part can be integrated out first:

$$\int_{-\infty}^{\infty} d\Omega \frac{(x)(i\Omega)e^{i\Omega t}}{(x^2+\Omega^2)(y^2+\Omega^2)} \sim \frac{e^{-xt}}{x^2-y^2} - \frac{e^{-yt}}{x^2-y^2} \tag{26}$$

We then have:

$$\frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} d^2k \frac{\vec{p}\cdot(\frac{\vec{p}}{2}-\vec{k})}{4(\frac{p^2}{4}+k^2+m^2)(\vec{p}\cdot\vec{k})\mu^2}\{e^{-\mu[(\frac{\vec{p}}{2}+\vec{k})^2+m^2]t} - e^{-\mu[(\frac{\vec{p}}{2}-\vec{k})^2+m^2]t}\}\mu[(\frac{\vec{p}}{2}+\vec{k})^2+m^2]$$

$$= \vec{p}\cdot\left(\frac{\vec{p}}{2}-\vec{k}\right)(-2)\frac{e^{-\mu(\frac{p^2}{4}+k^2+m^2)t}t(\vec{p}\cdot\vec{k})}{(\vec{p}\cdot\vec{k})} + \vec{p}\cdot\left(\frac{\vec{p}}{2}-\vec{k}\right)\frac{e^{-\mu(\frac{p^2}{4}+k^2+m^2)t}}{\frac{p^2}{4}+k^2+m^2}(-2)(\vec{p}\cdot\vec{k})t\ . \tag{27}$$

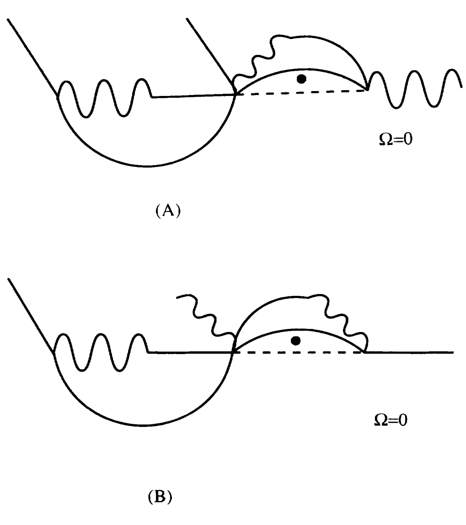

FIG. 18. Two Feynman diagrams contributing to $\Gamma_{1,2}$.

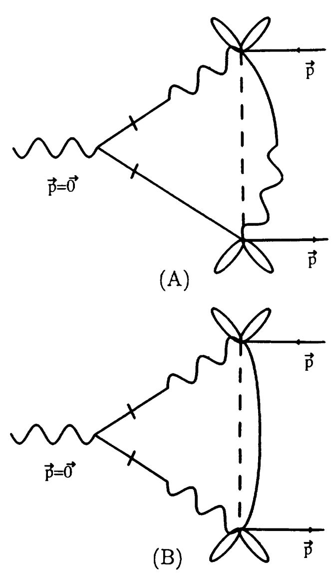

FIG. 19. The other two Feynman diagrams contributing to $\Gamma_{1,2}$.

In the hydrodynamic (long-wavelength) limit, $\vec{p} \to 0$. The relevant term in the first term in Eq. (27) can be easily found as

$$\int_{-\infty}^{\infty} d^2\vec{k}(-2)\frac{p^2}{2}te^{-\mu(\vec{k}^2+m^2)t} = -p^2\frac{1}{2\mu}e^{-\mu m^2 t}. \quad (28)$$

The relevant term in second term of Eq. (27) is

$$p^2(2)\int_0^{2\pi} d\theta \cos^2\theta \int_0^{\infty} kdkt\frac{k^2 e^{-\mu(k^2+m^2)t}}{k^2+m^2}$$

$$= p^2\frac{1}{2\mu}e^{-\mu m^2 t}. \quad (29)$$

Thus there are no contributions to the renormalization of λ due to their mutual cancellation. Other possible diagrams arise, but result in no contributions. In Fig. 18 those two diagrams will not contribute the renormalization when one imposes the long-time prescription ($\Omega_{ext} = 0$). The diagrams in Fig. 19 do not contribute either, since the interaction g is local in space and therefore there is no p-dependent part of the vertex $\Gamma_{1,2}$. To sum up, λ suffers no renormalization within the perturbative expansion to order g, and therefore $Z_\lambda = 1$.

VI. RECURSION RELATIONS

Once the Z factors are known to leading order in g, the recursion relations are obtained via the so-called β functions [25, 26, 29]:

$$\beta_\mu = \kappa\left(\frac{\partial\mu}{\partial\kappa}\right)_b = \mu\kappa\left(\frac{\partial \ln Z_{\tilde{\phi}}}{\partial\kappa}\right)_b, \quad (30)$$

$$\beta_D = \kappa\left(\frac{\partial D}{\partial\kappa}\right)_b = -D\kappa\left(\frac{\partial \ln Z_D}{\partial\kappa}\right)_b, \quad (31)$$

$$\beta_g = \kappa\left(\frac{\partial g}{\partial\kappa}\right)_b = -g\kappa\left(\frac{\partial \ln Z_g}{\partial\kappa}\right)_b, \quad (32)$$

$$\beta_{\bar{\nu}} = \kappa\left(\frac{\partial\bar{\nu}}{\partial\kappa}\right)_b, \quad (33)$$

where subscript b means that all bare parameters are fixed when one performs the differentiations [25, 26, 29] and κ is a mass scale. The renormalization of the couplings may also be related to the same β functions.

The renormalization Z factors are the ratios between the corresponding renormalized and bare parameters. Therefore it is a standard procedure to extract from their dependence on the momentum scale κ (or the bare mass m_0) the flow of the renormalized couplings under rescaling of all length scales by $\tilde{b} = \exp(l)$. The first step is to compute the so-called β functions, which when subtracted from the naive (engineering) dimension of the couplings yield the flow equations. Following this procedure, rescaling length scales $x \to \tilde{b}x$, momenta $k \to \tilde{b}^{-1}k$, time $t \to \tilde{b}^z t$, and frequencies $w \to \tilde{b}^{-z}w$, we find the following recursion relations:

$$\frac{d\nu}{dl} = 0, \quad (34)$$

$$\frac{d\bar{\nu}}{dl} = \frac{\pi\gamma^2}{4\nu(D\mu)^3}g^2, \quad (35)$$

$$\frac{dF}{dl} = 2F + \pi\lambda, \quad (36)$$

$$\frac{dD}{dl} = \left(\frac{\lambda^2}{8\pi}D\mu + \frac{\gamma^2\sqrt{c}g}{D\mu}\right)D, \quad (37)$$

$$\frac{d\mu}{dl} = \left(-\frac{\gamma^2\sqrt{c}g}{D\mu}\right)\mu, \quad (38)$$

$$\frac{dg}{dl} = \left(2 - \frac{D\mu\gamma^2}{2\pi} - \frac{\lambda^2 c'}{\gamma^2}\right)g - \frac{2\pi}{(D\mu)^2}g^2, \quad (39)$$

$$\frac{d\lambda}{dl} = 0. \quad (40)$$

γ is not renormalized because $Z_\phi = 1$ and keeps its bare value $\gamma = 2\pi$. For the same reason ν is not renormalized and may be chosen as $\nu = 1$.

The two constants are $c = \frac{1}{4}e^{2E} = 0.7931$ where E is the Euler constant and $c' \sim 180.08$, which is derived from the sum of the terms contributing to Z_g in Eq. (22).

VII. ASYMPTOTIC BEHAVIOR

In this chapter we proceed with the analysis of the physical implications of the recursion relations. We begin, in the next section by looking at the asymptotic destination of the flows which will yield the physical properties on very large scales of time and space. In the following subsection we will analyze the crossover behavior which determines the properties on intermediate scales.

A. Asymptotic behavior

The analysis of the recursion relations may be facilitated by the introduction of a "temperature" like variable (temperature is not well defined far away from equilibrium where the Einstein relation does not hold). Here we define it by $T = D\mu$ (it is not the thermodynamic temperature). Its recursion relation is obtained from Eqs. (37) and (38). It obeys

$$\frac{dT}{dl} = \frac{\lambda^2}{8\pi^2}T, \quad (41)$$

where the critical value $D\mu = 1/\pi$ is substituted. Since λ and γ are kept constant, this equation may be integrated:

$$T(l) = T_0 e^{\frac{\lambda^2}{8\pi^2}l}. \quad (42)$$

We see that no matter how small T_0 is $T(l)$ will grow indefinitely with $l = \ln b$ such that

$$T(l) = T_0\left(\frac{L}{a}\right)^{\lambda^2/8\pi^2}. \quad (43)$$

So the effective "temperature" becomes higher on longer length scales. Asymptotically the system is always at a high temperature. The growth of T is, however, quite slow. Therefore crossover effects discussed below play an important role.

What is the effect of high T? For that we have to look at the flow of g:

$$\frac{dg(l)}{dl} = \left[2 - \frac{T(l)}{2\pi}\gamma^2 - \frac{c'\lambda}{\gamma^2}\right] g(l) - \frac{\gamma^4 g^2}{8\pi^2}. \quad (44)$$

It is clear that if $T(l)$ grows very large it will cause $g(l)$ to decay to zero, no matter what are the bare values g_0, T_0, and λ. Once $g \to 0$ the asymptotic behavior becomes equivalent to that of the KPZ equation.

We thus conclude that asymptotically on very large scales and very long times the scaling properties are those associated with a far-from-equilibrium growth without the disorder and the periodic potential. The behavior will be determined by the effect of the lateral growth alone. The KPZ properties in $2+1$ dimensions (dominated by an inaccessible fixed point) will be the asymptotic ones for the system under consideration.

B. Crossover behavior

As we have found in the previous section, the temperature $T(l)$ rises with the scale quite slowly. Hence the decay of g to zero might also be slow. As long as g is not vanishing the effects of the disorder and the periodic potential are still felt. Hence we should expect a slowing down of the dynamics. This slowing down will be observable on larger scales as well because the mobility obeys the equation

$$\frac{\partial \mu}{\partial l} = -\frac{\gamma^2 \sqrt{c} g(l)}{T(l)} \mu(l), \quad (45)$$

and therefore

$$\mu(l) = \mu_0 e^{-\gamma^2 \sqrt{c} \int_0^l dl' [g(l')/T(l')]} . \quad (46)$$

The ratio $\mu(l)/\mu_0$ does depend on the integral

$$J(l) = \int_0^l \frac{g(l')}{T(l')} dl', \quad (47)$$

and clearly $J(l)$ is sensitive to $g(l)$ on small scales as well.

To evaluate $J(l)$ we need to know $T(l)$ given in Eq. (42) and $g(l)$, which we calculate next. Given $T(l)$, $g(l)$ is found by integration of its recursion relation:

$$\frac{1}{g(l)} = \frac{1}{g(0)} e^{s(l)} - \frac{\gamma^4}{8\pi^2} e^{s(l)} \int_0^l dx e^{-s(x)}, \quad (48)$$

where

$$s(x) = \left[\frac{\lambda^2 c'}{\gamma^2} - 2\right] x + T_0 \frac{\gamma^4}{\pi \lambda^2} (e^{\lambda^2 x/2\gamma^2} - 1). \quad (49)$$

Clearly the second term dominates for large x. Hence we see that for large l

$$g(l) \sim \exp\{-\exp(Al)\} , \quad (50)$$

with $A \sim \frac{\lambda^2}{8\pi^2}$.

It is easy to see that for large enough l, $g(l)$ decays to zero faster than exponentially. Since $T(l)$ diverges, the most important contribution to $J(l)$ comes from small (or at most intermediate) values of l. At large l, $J(l)$ asymptotically approaches a constant and its dependence on l becomes much weaker.

This asymptotic value of $J(l)$ depends mostly on the bare values of the parameters. To summarize, the mobility decays fast on initial small scales and then saturates to an almost constant value on large scales.

VIII. CONCLUSIONS

In this work we have investigated the behavior of a class of growth systems in which three different effects play important roles: periodicity, disorder, and lateral growth. Our model applies to the situations in which all three effects are present in deposition processes or solidification of rigid crystals.

In I we looked at the system near equilibrium when the lateral growth is negligible. There we found a continuous transition from a rough phase at high temperature into a super-rough and glassy phase for $T < T_g$.

The main conclusion of the present work is that far from equilibrium the KPZ term prevents this transition. The ultimate asymptotic behavior is dominated by the KPZ nonlinearity, while the second nonlinear term (obtained upon averaging the periodic potential over the disorder) is irrelevant.

We have seen, however, that while this term is decaying it still affects the behavior on intermediate scales. The effect of the disorder in the periodic potential is to slow the dynamics. In particular, the mobility decays from its bare value as

$$\frac{\mu(l)}{\mu(0)} = \exp[-4\pi^2 \times 1.78 J(l)], \quad (51)$$

where $J(l)$ is given by Eq. (47). It is clear that the behavior on intermediate scales drastically depends on the bare values of the parameters.

We have also shown that the theory is consistently renormalizable to first order in λ and g. To obtain the correct renormalization we had to keep diagrams up to and including three nontrivial loops. As far as we are aware, no other calculation has shown the renormalizability up to this order for a dynamic system for which the fluctuation-dissipation theorem is not satisfied. It is reassuring to see that the renormalization group can be successfully applied to the dynamics far from equilibrium.

ACKNOWLEDGMENTS

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APPENDIX A: TWO-LOOP CALCULATION FOR Z_g

As explained in Sec. IV, the calculation of Z_g is based on the evaluation of the vertex function $\Gamma_{2,0}$. For clarity, we neglect the prefactors and symmetry factors. Here we have six two-loop and two three-loop Feynman diagrams. The two-loop diagrams will be denoted by FD $2ln$, and three-loop diagrams will be denoted by FD $3ln$, where n stands for the sequel. First we look at the two-loop diagrams. The basic rules [26, 30] for the calculations of these diagrams have been described in the Appendix of our previous paper I. Here we shall simply write down the corresponding integral for each diagram. We employ the momentum-time representation for correlation and response functions, in terms of which the corresponding integrals will be easily handled.

Feynman diagram (FD) $2l1$ is shown in Fig. 8. The corresponding integral over time is given by

$$I_{\mathrm{FD}\ 2l1} = \int_0^\infty dt_y \int_0^{t_y} dt_x \frac{[\vec{q}\cdot(\vec{q}-\vec{p})](\vec{p}\cdot\vec{q})}{(q^2+m^2)(p^2+m^2)} e^{-[q^2+(\vec{p}-\vec{q})^2+2m^2]t_x} e^{-(p^2+m^2)t_y} e^{-(p^2+m^2)(t_y-t_x)}. \tag{A1}$$

The integration of the time dependent sectors gives

$$\begin{aligned}\int_0^\infty dt_y \int_0^{t_y} & dt_x e^{-[q^2+(\vec{p}-\vec{q})^2+m^2-p^2]t_x} e^{-2(p^2+m^2)t_y} \\ &= \int_0^\infty dt_y e^{-2(p^2+m^2)t_y} \frac{-1}{[q^2+(\vec{p}-\vec{q})^2+m^2-p^2]}[e^{-[q^2+(\vec{p}-\vec{q})^2+m^2]t_y}-1] \\ &= \frac{1}{2(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}.\end{aligned} \tag{A2}$$

By decomposing $\vec{p}\cdot\vec{q}$ into

$$\vec{p}\cdot\vec{q} = \frac{-1}{2}[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2-2p^2-2q^2-3m^2] \tag{A3}$$

and $\vec{q}\cdot(\vec{q}-\vec{p})$ into

$$(q^2-\vec{p}\cdot\vec{q}) = \frac{1}{2}[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2-2p^2-3m^2]\ , \tag{A4}$$

we obtain

$$\begin{aligned}&\frac{(\vec{q}\cdot\vec{p})}{(p^2+m^2)(q^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]} \\ &\quad = -\frac{1}{2}\left(\frac{1}{(p^2+m^2)(q^2+m^2)} - \frac{2}{(p^2+m^2)(q^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}\right. \\ &\qquad \left. - \frac{2}{(p^2+m^2)^2[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]} + \text{irrelevant terms}\right).\end{aligned} \tag{A5}$$

Since we impose the minimal subtraction (MS) [25, 29] scheme, the unwanted nonsingular parts (finite parts) will be ignored in all calculations in this paper. In this entire paper, an irrelevant term means a term which does not contribute to the singular part of the integral. We sometimes use an equal sign to represent the equality of the singular parts on both sides.

We substitute Eqs. (A4) and (A5) into Eq. (A1), and obtain

$$\begin{aligned}I_{\mathrm{FD}\ 2l1} &= -\frac{1}{4}\left\{\frac{(q^2-\vec{p}\cdot\vec{q})}{(p^2+m^2)^2(q^2+m^2)} - \frac{[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2-2p^2-3m^2]}{(p^2+m^2)(q^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}\right. \\ &\qquad \left. - \frac{[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2-2p^2-3m^2]}{(p^2+m^2)^2[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}\right\} \\ &= -\frac{1}{4}\left[\frac{1}{(p^2+m^2)^2} + \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)} - \frac{1}{(p^2+m^2)(q^2+m^2)}\right. \\ &\qquad \left. + \frac{2}{(q^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]} - \frac{1}{(p^2+m^2)^2} + \frac{2}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}\right] \\ &= [\tfrac{1}{4}\tilde{B} - \tilde{C}],\end{aligned} \tag{A6}$$

where

$$\tilde{B}=\frac{1}{(p^2+m^2)(q^2+m^2)}\ , \tag{A7}$$

$$\tilde{C}=\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}, \tag{A8}$$

and the second term in Eq. (A6) is discarded because it vanishes after the integration over the momentum variables.

In the same way, the associated integral for Fig. 9 reads

$$I_{\mathrm{FD}\ 212}=\int_0^\infty dt_x\int_{-\infty}^{t_x}dt_y\frac{[\vec{q}\cdot(\vec{p}-\vec{q})]}{(q^2+m^2)}\frac{[\vec{q}\cdot(\vec{p}-\vec{q})]}{[(\vec{p}-\vec{q})^2+m^2]}e^{-(p^2+m^2)t_x}e^{-[(\vec{p}-\vec{q})^2+q^2+2m^2]|t_y|}e^{-(p^2+m^2)(t_x-t_y)}. \tag{A9}$$

To begin with, we integrate over the time variables, and that yields

$$\int_0^\infty dt_x\int_{-\infty}^0 dt_y e^{-2(p^2+m^2)t_x}e^{(p^2+m^2)t_y+[(\vec{p}-\vec{q})^2+q^2+2m^2]t_y}$$

$$+\int_0^\infty dt_x\int_0^{t_x}dt_y e^{-2(p^2+m^2)t_x}e^{(p^2+m^2)t_y-[(\vec{p}-\vec{q})^2+q^2+2m^2]t_y}=\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}. \tag{A10}$$

By inserting Eq. (A10) into Eq. (A9), we obtain

$$\frac{[\vec{q}\cdot(\vec{p}-\vec{q})][\vec{q}\cdot(\vec{p}-\vec{q})]}{(q^2+m^2)(p^2+m^2)[(\vec{p}-\vec{q})^2+m^2][p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}$$

$$=\frac{3}{4}\frac{q^2}{(p^2+m^2)[(\vec{p}-\vec{q})^2+m^2][p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}-\frac{1}{2}\frac{1}{[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2](q^2+m^2)}$$
$$=[\tfrac{3}{4}\tilde{A}-\tfrac{1}{2}\tilde{C}], \tag{A11}$$

where

$$\tilde{A}=\frac{q^2}{(p^2+m^2)[(\vec{p}-\vec{q})^2+m^2][p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}. \tag{A12}$$

For the Feynman diagram in Fig. 10, the integral with the time variables is written as

$$I_{\mathrm{FD}\ 213}=\frac{(\vec{q}\cdot\vec{p})(\vec{p}\cdot\vec{p})}{(p^2+m^2)(q^2+m^2)}\int_0^\infty dt_x\int_0^{t_x}dt_y e^{-(p^2+m^2)(t_x-t_y)}e^{[(\vec{p}-\vec{q})^2+q^2+2m^2]t_y}. \tag{A13}$$

Again, we integrate over the time variables first, and obtain

$$\int_0^\infty dt_x\int_0^{t_x}dt_y e^{-2(p^2+m^2)t_x}e^{-[q^2+(\vec{p}-\vec{q})^2+2m^2]t_y+(p^2+m^2)t_y}=\frac{1}{2}\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}. \tag{A14}$$

With the help of Eq. (A14), Eq. (A13) is simplified to

$$\frac{(\vec{p}\cdot\vec{q})(\vec{p}\cdot\vec{q})}{(p^2+m^2)(q^2+m^2)}\frac{1}{2}\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}=\frac{1}{4}[4\tilde{C}-\tilde{B}]. \tag{A15}$$

Now we turn to the diagram in Fig. 11. The associated integral is represented by

$$I_{\mathrm{FD}\ 214}=-\frac{p^2[(\vec{p}-\vec{q})\cdot\vec{q}]}{(q^2+m^2)[(\vec{p}-\vec{q})^2+m^2]}\int_0^\infty dt_x\int_{-\infty}^{t_x}dt_y e^{-(p^2+m^2)t_x}e^{-[q^2+(\vec{p}-\vec{q})^2+2m^2]|t_y|}e^{-(p^2+m^2)(t_x-t_y)}. \tag{A16}$$

The time dependent sector in Fig. 11 is identical to Eq. (A10), so we will not repeat the calculation here. In the same fashion, the integrand takes the form

$$\frac{[q^2-\vec{p}\cdot\vec{q}]}{(q^2+m^2)[(\vec{p}-\vec{q})^2+m^2][p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}=[\tfrac{1}{2}\tilde{B}-\tilde{A}]. \tag{A17}$$

For the diagram in Fig. 12, we have the associated integral as

$$I_{\rm FD\ 2l5} = \frac{-[\vec{q}\cdot(\vec{p}-\vec{q})](p^2)}{(q^2+m^2)(p^2+m^2)}\int_0^\infty dt_x \int_0^{t_x} dt_y e^{-[q^2+(\vec{p}-\vec{q})^2+2m^2]t_y} e^{-(p^2+m^2)(t_x-t_y)} e^{-(p^2+m^2)t_x}. \quad \text{(A18)}$$

The time dependent term is integrated out first:

$$\int_0^\infty dt_x \int_0^{t_x} dt_y e^{-[q^2+(\vec{p}-\vec{q})^2+2m^2]t_y} e^{-(p^2+m^2)(t_x-t_y)} e^{-(p^2+m^2)t_x} = \frac{1}{2}\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]}. \quad \text{(A19)}$$

Then we obtain

$$-\frac{[\vec{q}\cdot(\vec{p}-\vec{q})](p^2)}{(q^2+m^2)(p^2+m^2)}\frac{1}{2}\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]} = \frac{1}{4}(\tilde{B}-2\tilde{C}). \quad \text{(A20)}$$

We turn to the simplest figure among two-loop diagrams, Fig. 13, of which the related integral is evaluated as

$$\left\{\int_{-\infty}^\infty \int_0^\infty dt p^2 R_0(\vec{p},t) C_0(\vec{p},t)\right\}^2 = \frac{1}{4}\left[\int_{-\infty}^\infty d^d\vec{p}\frac{1}{(p^2+m^2)}\right]^2, \quad \text{(A21)}$$

$$I_{\rm FD\ 2l6} = -\frac{1}{4}\int_{-\infty}^\infty d^d\vec{p}\frac{1}{(p^2+m^2)}\int_{-\infty}^\infty d^d\vec{p}\frac{1}{(p^2+m^2)} = -\frac{1}{4}B. \quad \text{(A22)}$$

Now we go on to perform the integration over the momentum variables. Before the evaluation of the singular parts of the integrals, it will be helpful to present some identities, which will play important roles in later calculations and have been frequently employed in these types of calculations. The first one is the Feynman parametrization formula, which reads

$$\frac{1}{A^\alpha B^\beta \cdots E^\sigma} = \frac{\Gamma(\alpha+\beta+\gamma+\cdots+\epsilon)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)\cdots\Gamma(\sigma)}\int_0^1 \cdots \int_0^1 dx dy dz \cdots \delta(1-x-y-z\cdots)\frac{x^{\alpha-1}y^{\beta-1}\cdots z^{\sigma-1}}{(Ax+By+\cdots+Ez)^{\alpha+\beta+\gamma+\cdots+\sigma}}. \quad \text{(A23)}$$

A set of integral formulas is also valuable and is shown as

$$J_0 = \int_{-\infty}^\infty d^d\vec{k}\frac{1}{(\vec{k}^2+2\vec{k}\cdot\vec{p}+M)^\alpha} = \frac{\pi^{d/2}}{\Gamma(\alpha)}(M-p^2)^{d/2-\alpha}\Gamma\left(\alpha-\frac{d}{2}\right), \quad \text{(A24)}$$

$$\int_{-\infty}^\infty d^d\vec{k}\frac{k^\nu}{(\vec{k}^2+2\vec{k}\cdot\vec{p}+M)^\alpha} = -p^\nu J_0. \quad \text{(A25)}$$

With these formulas, one can evaluate the singular parts of the integrals. Let $X = \int\int d^d\vec{p}d^d\vec{q}\tilde{X}$, where $X = A$, B, or C, and $d = 2+\epsilon$. The evaluations of B, C, and A are carried out as

$$B^{1/2} = \int_{-\infty}^\infty d^d\vec{p}\frac{1}{(p^2+m^2)} = \pi^{d/2}(m^2)^{\frac{d}{2}-1}\frac{\Gamma(1-\frac{d}{2})}{\Gamma(1)}, \quad \text{(A26)}$$

$$\begin{aligned} C &= \int_{-\infty}^\infty d^d\vec{p}\int_{-\infty}^\infty d^d\vec{q}\frac{1}{(p^2+m^2)[p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]} \\ &= \frac{1}{2}\int_{-\infty}^\infty d^d\vec{p}\int_{-\infty}^\infty d^d\vec{q}\frac{1}{(p^2+m^2)}\frac{1}{(p^2+q^2-\vec{p}\cdot\vec{q}+\frac{3}{4}m^2)} + \text{nonsingular terms} \\ &= \frac{1}{2}\int_{-\infty}^\infty d^d\vec{p}\frac{1}{(p^2+m^2)}\pi^{d/2}\frac{\Gamma(1-d/2)}{\Gamma(1)}\frac{1}{(\frac{3}{4}p^2+\frac{3}{4}m^2)^{-\epsilon/2}} \\ &= \frac{1}{2}\pi^d\frac{\Gamma(-\epsilon/2)\Gamma(-\epsilon)}{\Gamma(1-\epsilon/2)}\left(\frac{3}{4}\right)^{\epsilon/2} m^{2\epsilon}, \end{aligned} \quad \text{(A27)}$$

$$\begin{aligned} A &= \int_{-\infty}^\infty d^d\vec{p}\int_{-\infty}^\infty d^d\vec{q}\frac{q^2}{(p^2+m^2)[(\vec{p}-\vec{q})^2+m^2][p^2+q^2+(\vec{p}-\vec{q})^2+3m^2]} \\ &= \int_{-\infty}^\infty d^d\vec{p}\int_{-\infty}^\infty d^d\vec{q}\frac{q^2}{(p^2+m^2)[(\vec{p}-\vec{q})^2+m^2][p^2+q^2+(\vec{p}-\vec{q})^2+2m^2]} \end{aligned}$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{\Gamma(3)q^2}{\{(1-x-y)(p^2+m^2)+x[(\vec{p}-\vec{q})^2+m^2]+y(p^2+q^2-\vec{p}\cdot\vec{q}+m^2)\}^3}$$

$$= \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{q^2}{[p^2-\vec{p}\cdot\vec{q}(2x+y)+(x+y)q^2+m^2]^3}$$

$$= \int_{-\infty}^{\infty} d^d\vec{q}q^2 \int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{\Gamma(3-d/2)}{\Gamma(3)}(\pi)^{d/2} \frac{1}{\{[(x+y)-(x+\frac{y}{2})^2]q^2+m^2\}^{2-\epsilon/2}}$$

$$= (\pi)^{d/2}\frac{\Gamma(3-d/2)}{\Gamma(3)} \int_0^1 \int_{0\ x+y<1}^1 dxdy \int_{-\infty}^{\infty} d^d\vec{q} \frac{q^2}{[(x+y)-(x+\frac{y}{2})^2]^{2-\epsilon/2}} \frac{1}{\left\{q^2+\frac{m^2}{[(x+y)-(x+\frac{y}{2})^2]}\right\}^{2-\epsilon/2}}$$

$$= (\pi)^{d/2}\frac{\Gamma(3-d/2)}{\Gamma(3)} \int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{1}{[(x+y)-(x+\frac{y}{2})^2]^{2-\epsilon/2}}$$

$$\times \left\{ m\frac{1}{[(x+y)-(x+\frac{y}{2})^2]^{1/2}} \right\}^{d+2-2(2-\epsilon/2)} (\pi)^{d/2}\frac{\Gamma(1+d/2)}{\Gamma(d/2)} \frac{\Gamma(-\epsilon)}{\Gamma(2-\epsilon/2)}$$

$$= (\pi)^d \frac{\Gamma(2+\epsilon/2)}{\Gamma(1+\epsilon/2)} \frac{\Gamma(-\epsilon)}{2} \int_0^1 \int_{0\ x+y<1}^1 dxdy m^{2\epsilon} \frac{1}{[(x+y)-(x+\frac{y}{2})^2]^{2+\epsilon/2}}. \quad \text{(A28)}$$

The singular parts of the integral in Eq. (A28) can be evaluated by the changes of variables, $x = st$, $y = s(1-t)$. Let A' denote the x, y dependent part in Eq. (A28). We obtain

$$A' = \int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{1}{[(x+y)-(x+\frac{y}{2})^2]^{2+\epsilon/2}}$$

$$= \int_0^1 \int_0^1 dsdt \frac{s}{[s-s^2(1-\frac{t}{2})]^{2+\epsilon/2}}$$

$$= \int_0^1 \int_0^1 dsdt \frac{1}{s^{1+\epsilon/2}[1-(1-\frac{t}{2})s]^{2+\epsilon/2}}$$

$$= \int_0^1 \int_0^1 dsdt \frac{1}{s^{1+\epsilon/2}} \left\{ \frac{1}{[1-(1-\frac{t}{2}s)]^{2+\epsilon/2}} - 1 \right\} + \int_0^1 ds \frac{1}{s^{1+\epsilon/2}}. \quad \text{(A29)}$$

From the form of Eq. (A29), it is not hard to see that the singular term as $\epsilon \to 0$ is the manifestation of the singular behavior of the pole $s = 0$ in the integral. To simplify the expressions of the equations, the term $1 - \frac{t}{2}$ is symbolized by α. Equation (A29) can be rewritten as

$$A' = \int_0^1 \int_0^1 dsdt \frac{1}{s^{1+\epsilon/2}} \frac{2\alpha s - \alpha^2 s^2}{(1-\alpha s)^{2+\epsilon/2}} + \int_0^1 ds \frac{1}{s^{1+\epsilon/2}}. \quad \text{(A30)}$$

The analysis of two terms in Eq. (A30) will be carried out in order. The first term in Eq. (A30) is recast into

$$\int_0^1 \int_0^1 dsdt \frac{1}{s^\epsilon} \frac{\alpha}{(1-\alpha s)^{2+\epsilon/2}} + \int_0^1 \int_0^1 dsdt \frac{1}{s^\epsilon} \frac{\alpha}{(1-\alpha s)^{1+\epsilon/2}}$$

$$= \int_0^1 \int_0^1 dsdt \frac{\alpha}{(1-\alpha s)^{2+\epsilon/2}} + \epsilon \int_0^1 \int_0^1 dsdt \frac{[\ln(s)]\alpha}{(1-\alpha s)^{2+\epsilon/2}} + \int_0^1 \int_0^1 dsdt \frac{\alpha}{(1-\alpha s)^{1+\epsilon/2}} + O(\epsilon). \quad \text{(A31)}$$

The second term in Eq. (A31) is of order ϵ and therefore is discarded. The first term in Eq. (A31) is evaluated as

$$\int_0^1 \int_0^1 dsdt \frac{\alpha}{(1-\alpha s)^{2+\epsilon/2}} = \frac{1}{1+\epsilon/2} \left\{ \int_0^1 dt \left[\frac{1}{[1-\alpha]^{1\epsilon+/2}} - 1 \right] \right\}$$

$$= \frac{1}{1+\epsilon/2} \left\{ \int_0^1 dt \frac{1}{(t-\frac{t^2}{4})^{1+\epsilon/2}} - 1 \right\}, \quad \text{(A32)}$$

where

$$\begin{aligned}\int_0^1 dt \frac{1}{(t-\frac{t^2}{4})^{1+\epsilon/2}} &= \int_0^1 dt \frac{1}{t^{1+\epsilon/2}} \left[\frac{1}{(1-\frac{t}{4})^{1+\epsilon/2}} - 1\right] + \int_0^1 dt \frac{1}{t^{1+\epsilon/2}} \\ &= \int_0^1 dt \frac{1}{t^{1+\epsilon/2}} \frac{\frac{t}{4} + (1-\frac{t}{4})\frac{\epsilon}{2}\ln(1-\frac{t}{4}) + \cdots}{(1-\frac{t}{4})^{1+\epsilon/2}} + \int_0^1 dt \frac{1}{t^{1+\epsilon/2}} \\ &= \int_0^1 dt \frac{1}{4t^{1+\epsilon/2}} \frac{1}{(1-\frac{t}{4})^{1+\epsilon/2}} + O(\epsilon) + \frac{1}{-\epsilon/2} \\ &= \frac{1}{4}\int_0^1 dt \frac{1}{(1-\frac{t}{4})} - \frac{2}{\epsilon} + O(\epsilon) \\ &= \ln(\tfrac{4}{3}) - \frac{2}{\epsilon} + O(\epsilon). \end{aligned} \tag{A33}$$

The third term in Eq. (A31) is

$$\begin{aligned}\int_0^1\int_0^1 dsdt \frac{\alpha}{(1-\alpha s)^{1+\epsilon/2}} &= \int_0^1\int_0^1 dsdt \frac{\alpha}{(1-\alpha s)} + O(\epsilon) \\ &= -\int_0^1 dt \ln\left[1 - \left(1-\frac{t}{2}\right)^2\right] + O(\epsilon) \\ &= 3\ln(\tfrac{4}{3}) + O(\epsilon)\ . \end{aligned} \tag{A34}$$

The second term in Eq. (A30) equals $-\frac{2}{\epsilon}$.

With the combination of the prefactors in Eq. (A28) and the results obtained above, the singular part of A is given by

$$A = (\pi)^d \left[\frac{2}{\epsilon^2} - \frac{2}{\epsilon}\ln(\tfrac{4}{3}) - \frac{2\gamma}{\epsilon} + \frac{1}{\epsilon}\right] + \text{(finite terms)}, \tag{A35}$$

where γ is the Euler number. The finite part of the subdivergence diagram can be neglected, if the scale equation is well defined. One can always scale it away [25]. In the short distance cutoff [25, 28], they appear as

$$A = \frac{1}{16\pi^2}\{\tfrac{1}{2}[\ln(cm^2a^2)]^2 - \ln(\tfrac{4}{3})\ln(cm^2a^2) + \tfrac{1}{2}\ln(cm^2a^2)\}\ , \tag{A36}$$

$$B = \frac{1}{16\pi^2}\{[\ln(cm^2a^2)]^2\}\ , \tag{A37}$$

$$C = \frac{1}{16\pi^2}\{\tfrac{1}{4}[\ln(cm^2a^2)]^2 + \tfrac{1}{4}\ln(\tfrac{3}{4})\ln(cm^2a^2)\}\ . \tag{A38}$$

TABLE I. Symmetry factors and integrals of two-loop diagrams.

Feynman diagrams	Symmetry factors	Integrals
FD 2l1	16	$(\frac{1}{4}B - C)$
FD 2l2	8	$(\frac{3}{4}A - \frac{1}{2}C)$
FD 2l3	32	$(C - \frac{1}{4}B)$
FD 2l4	16	$(\frac{1}{2}B - A)$
FD 2l5	32	$(\frac{1}{4}B - \frac{1}{2}C)$
FD 2l6	16	$-\frac{1}{4}B$

Now we retrieve the symmetry factors for each two-loop diagrams (see Table I). As one can see from Figs. 8–13, only Figs. 9, 12, and 13 contain subdivergent diagrams. One also can verify this from the results listed in Table I. The leading terms in the diagrams without subdivergences, such as Figs. 8, 10, and 11, are of order $\ln(cm^2a^2)$. On the other hand, the leading terms of the diagrams with subdivergence as mentioned above are of order $[\ln(cm^2a^2)]^2$. Furthermore, the leading divergences of the diagrams in Figs. 12 and 13 are canceled out by each other. They have the same type of subdivergences (see the subdiagrams enclosed by the boxes in their own figures), which are not present in the lower-order (one-loop) calculation. Another diagram in Fig. 9 includes the subdivergent diagram (see the subdiagram enclosed by the box in Fig. 9), which occurs in the one-loop calculation for D. As usual, it will be canceled when one calculates the recursion relations, even though the Z_g factors contain some terms like $[\ln(cm^2a^2)]^2$. As we will show in Appendix B, the three-loop diagrams do not consist of any subdivergent diagrams. Thus, the subdivergence in the present expansion only occurs in two-loop diagrams. The cancellation of subdivergences in Figs. 12 and 13, and that of Figs. 9 and 3, ensure the renormalizability of this theory (at least at this order). Leading divergences and subleading divergences are summed up to contribute to Z_g [see the third and fourth terms in Eq. (22)].

APPENDIX B: THREE-LOOP CALCULATION FOR Z_g

In this appendix, we shall present the calculations for three-loop diagrams, which are mentioned in Sec. IV. The diagram shown in Fig. 14, representing the integral, is given by

$$I_{\rm FD\ 3l1} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{-[\vec{p}\cdot\vec{q}][(\vec{p}-\vec{k})\cdot\vec{k}]}{(q^2+m^2)[(\vec{p}-\vec{k})^2+m^2](k^2+m^2)}$$
$$\times \int_0^{\infty} dt_y \int_0^{t_y} dt_x e^{-\{(q^2+m^2)+[(\vec{p}-\vec{q})^2+m^2]\}(t_y-t_x)} e^{-(p^2+m^2)t_x} e^{-\{(k^2+m^2)+[(\vec{p}-\vec{k})^2+m^2]\}t_y}. \quad \text{(B1)}$$

To simplify the calculation, we denote $a=(q^2+m^2)$, $b=[(\vec{p}-\vec{q})^2+m^2]$, $c=(k^2+m^2)$, $d=[(\vec{p}-\vec{k})^2+m^2]$, and $e=(p^2+m^2)$. Along the same line as our preceding calculations of two-loop diagrams, we integrate out the time dependent term first:

$$\int_0^{\infty} dt_y e^{-\{(q^2+m^2)+[(\vec{p}-\vec{q})^2+m^2]+(k^2+m^2)+[(\vec{p}-\vec{k})^2+m^2]\}t_y} \int_0^{t_y} dt_x \quad e^{-\{(p^2+m^2)-(q^2+m^2)-[(\vec{p}-\vec{q})^2+m^2]\}t_x}$$

$$= \int_0^{\infty} dt_y e^{-(a+b+c+d)t_y}[e^{-(e-a-b)t_y}-1]\frac{1}{b+c-e} = \int_0^{\infty} dt_y \frac{1}{b+c-a}\{e^{-(e+c+d)t_y}-e^{-(a+b+c+d)t_y}\}$$
$$= \frac{1}{(c+d+e)(a+b+c+d)}. \quad \text{(B2)}$$

Therefore

$$I_{\rm FD\ 3l1} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{-[\vec{p}\cdot\vec{q}][\vec{p}-\vec{k}]\cdot\vec{k}}{(q^2+m^2)[(\vec{p}-\vec{k})^2+m^2](k^2+m^2)} \frac{1}{(c+d+e)(a+b+c+d)}$$
$$= \left(\frac{-1}{2}\right) \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)(k^2+m^2)[(\vec{p}-\vec{k})^2+m^2]}$$
$$\times \frac{1}{\{(q^2+m^2)+(k^2+m^2)+[(\vec{p}-\vec{q})^2+m^2]+[(\vec{p}-\vec{k})^2+m^2]\}}$$
$$- \frac{2\vec{p}\cdot\vec{q}}{(q^2+m^2)[(\vec{p}-\vec{k})^2+m^2]\{(k^2+m^2)+(p^2+m^2)+[(\vec{p}-\vec{k})^2+m^2]\}}$$
$$\times \frac{1}{\{(q^2+m^2)+(k^2+m^2)+[(\vec{p}-\vec{q})^2+m^2]+[(\vec{p}-\vec{k})^2+m^2]\}}$$
$$- \frac{2\vec{p}\cdot\vec{q}}{(q^2+m^2)(k^2+m^2)[(p^2+m^2)+(k^2+m^2)+(\vec{p}-\vec{k})^2+m^2]}$$
$$\times \frac{1}{\{(q^2+m^2)+(k^2+m^2)+[(\vec{p}-\vec{q})^2+m^2]+[(\vec{p}-\vec{k})^2+m^2]\}}, \quad \text{(B3)}$$

where we have used the identities below to simplify the expression of the equation:

$$\frac{\vec{s}\cdot\vec{k}}{c+d+e} = \frac{1}{2}\left[1 - \frac{-2k^2-2s^2-3m^2}{c+d+e}\right]\frac{1}{a+b+c+d}, \quad \text{(B4)}$$

where $\vec{s}=\vec{p}-\vec{k}$. To simplify the calculation, we treat Eq. (B3) as a linear combination of three integrals, $I_{3.1-1}$, $I_{3.1-2}$, and $I_{3.1-3}$. Namely,

$$I_{\rm FD\ 3l1} = \frac{-1}{2}(I_{3.1-1} - 2I_{3.1-2} - 2I_{3.1-3}). \quad \text{(B5)}$$

The term denoted by $I_{3.1-1}$ yields

$$I_{3.1-1} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)(k^2+m^2)[(\vec{p}-\vec{k})^2+m^2]}$$
$$\times \frac{1}{\{(q^2+m^2)+(k^2+m^2)+[(\vec{p}-\vec{q})^2+m^2]+[(\vec{p}-\vec{k})^2+m^2]\}}$$
$$= \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \frac{1}{2} \frac{\Gamma(1+1+1)}{\Gamma(1)\Gamma(1)\Gamma(1)} \int_0^1 \int_{0\ \ x+y<1}^1 dx dy \int_{-\infty}^{\infty} d^d\vec{k}$$
$$\times \frac{\vec{p}\cdot\vec{q}}{[k^2+\vec{k}\cdot\vec{p}(-2x-y)+p^2(x+y)+\vec{p}\cdot\vec{q}(-y)+yq^2+m^2]^3}$$

$$= (\pi)^{d/2}\frac{\Gamma(2-\epsilon/2)}{\Gamma(3)}\frac{\Gamma(3)}{2}\int_{-\infty}^{\infty}d^d\vec{p}\int_{-\infty}^{\infty}d^d\vec{q}\int_0^1\int_{0\; x+y<1}^1 dxdy$$
$$\times\frac{\vec{p}\cdot\vec{q}}{\{[(x+y)-(x+\frac{y}{2})^2]p^2+\vec{p}\cdot\vec{q}(-y)+yq^2+m^2\}^{2-\epsilon/2}}$$
$$= (\pi)^{d/2}\frac{\Gamma(2-\epsilon/2)}{2}\int_{-\infty}^{\infty}d^d\vec{p}\int_{-\infty}^{\infty}d^d\vec{q}\int_0^1\int_{0\; x+y<1}^1 dxdy\frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)y^{2-\epsilon/2}}\frac{1}{[q^2-\vec{p}\cdot\vec{q}+\frac{\Delta}{y}p^2+\frac{1}{y}m^2]^{2-\epsilon/2}}$$
$$= (\pi)^{d/2}\frac{\Gamma(2-\epsilon/2)}{2}\int_{-\infty}^{\infty}d^d\vec{p}\int_{-\infty}^{\infty}d^d\vec{q}\int_0^1\int_0^1\int_{0\; x+y<1}^1 dxdydz$$
$$\times\frac{\Gamma(3-\epsilon/2)}{\Gamma(1)\Gamma(2-\epsilon/2)}\frac{z^{1-\epsilon/2}\quad \vec{p}\cdot\vec{q}}{y^{2-\epsilon/2}[q^2-z\vec{p}\cdot\vec{q}+\frac{z\Delta}{y}p^2+\frac{z}{y}m^2]^{3-\epsilon/2}}$$
$$= (\pi)^{d/2}\frac{\Gamma(2-\epsilon/2)\Gamma(3-\epsilon/2)}{\Gamma(2-\epsilon/2)}\frac{z^{1-\epsilon/2}}{y^{2-\epsilon/2}}\frac{\frac{z}{2}p^2}{\{[\frac{z\Delta}{y}-(\frac{z}{2})^2]p^2+\frac{z}{y}m^2\}^{2-\epsilon}}(\pi)^{d/2}\frac{\Gamma(2-\epsilon)}{\Gamma(3-\epsilon/2)}$$
$$= \frac{(\pi)^d\Gamma(2-\epsilon)}{2\times 2}\int_0^1\int_0^1\int_{0\; x+y<1}^1 dxdydz\int_{-\infty}^{\infty}d^d\vec{p}\frac{z^{2-\epsilon/2}}{y^{2-\epsilon/2}(\Lambda)^{2-\epsilon}}\frac{p^2}{[p^2+\frac{zm^2}{y\Lambda}]^{2-\epsilon}}$$
$$= \frac{\pi^d\Gamma(2-\epsilon)}{4}\int_0^1\int_0^1\int_{0\; x+y<1}^1 dxdydz\frac{z^{2-\epsilon/2}}{y^{2-\epsilon/2}\Lambda^{2-\epsilon}}\left[m\left(\frac{z}{y\Lambda}\right)^{1/2}\right]^{3\epsilon}(\pi)^{d/2}\frac{\Gamma(2+\epsilon/2)}{\Gamma(1+\epsilon/2)}\frac{\Gamma(-3/2\epsilon)}{\Gamma(2-\epsilon)}$$
$$= \frac{\pi^{3/2d}}{4}\frac{\Gamma(2+\epsilon/2)\Gamma(-3/2\epsilon)}{\Gamma(1+\epsilon/2)}(m^2)^{3\epsilon/2}\int_0^1\int_0^1\int_{0\; x+y<1}^1 dxdydz\frac{z^{2+\epsilon}}{y^{2+\epsilon}(\Lambda)^{2+\epsilon/2}}, \tag{B6}$$

where $\Delta=(x+y)-(x-\frac{y}{2})^2$ and $\Lambda=\frac{z}{y}\Delta-\frac{z^2}{4}$.

The working principles of extracting the singularity of the integral is based on the separation of the singular contributions from different points. All the calculations here follow this scenario. However, one should be able to keep track of those highly nested procedures. In the last line of Eq. (B6), the prefactors before the integral contain a leading singular pole of order $\frac{1}{\epsilon}$, and thus we should extract the contribution up to the zero order in ϵ from this integral. The extraction of the poles and finite parts of this integral proceeds as

$$\int_0^1\int_0^1\int_{0\; x+y<1}^1 dxdydz\frac{z^{2+\epsilon}}{y^{2+\epsilon}\Lambda^{2+\epsilon/2}}=\int_0^1 dz\int_0^1\int_{0\; x+y<1}^1 dxdy\frac{z^{\epsilon/2}}{y^{\epsilon/2}\{[(x+y)-(x+\frac{y}{2})^2]-\frac{zy}{4}\}^{2+\epsilon/2}}. \tag{B7}$$

Rewriting x,y as $y=st, x=s(1-t)$, we inherit the simplified equation

$$\text{Eq. (B7)}=\int_0^1 dz\int_0^1\int_0^1 dsdt\frac{sz^{\epsilon/2}}{s^{\epsilon/2}t^{\epsilon/2}[s-s^2(1-\frac{t}{2})^2-\frac{z}{4}st]^{2+\epsilon/2}}$$
$$=\int_0^1 dz\int_0^1\int_0^1 dsdt\frac{z^{\epsilon/2}}{t^{\epsilon/2}}\frac{1}{s^{1+\epsilon}[1-s(1-\frac{t}{2})^2-\frac{zt}{4}]^{2+\epsilon/2}}. \tag{B8}$$

Recast Eq. (B8) into

$$\text{Eq. (B8)}=\int_0^1 dz\int_0^1\int_0^1 dsdt\frac{z^{\epsilon/2}}{t^{\epsilon/2}}\frac{1}{(1-\frac{zt}{4})^{2+\epsilon/2}s^{1+\epsilon}(1-s\tilde{\alpha})^{2+\epsilon/2}}, \tag{B9}$$

where $\tilde{\alpha}=\frac{(1-t/2)^2}{(1-zt/4)}$. Consider the integration over s first:

$$\int_0^1 ds\frac{1}{s^{1+\epsilon}(1-s\tilde{\alpha})^{2+\epsilon/2}}=\int_0^1 ds\frac{s\tilde{\alpha}[2-s\tilde{\alpha}]+[1-s\tilde{\alpha}]^2\frac{\epsilon}{2}\ln(1-s\tilde{\alpha})+\cdots}{s^{1+\epsilon}(1-s\tilde{\alpha})^{2+\epsilon/2}}+\int_0^1 ds\frac{1}{s^{1+\epsilon}}=A_1+A_2. \tag{B10}$$

A_1 can be represented as the sum of A_{11} and A_{12}.

$$A_1=A_{11}+A_{12}=\int_0^1 ds\frac{\tilde{\alpha}}{s^{\epsilon}(1-s\tilde{\alpha})^{2+\epsilon/2}}+\int_0^1 ds\frac{\tilde{\alpha}}{s^{\epsilon}(1-s\tilde{\alpha})^{1+\epsilon/2}}. \tag{B11}$$

Furthermore, the term A_{11} can be decomposed into the following:

$$A_{11}=A_{111}+A_{112}=\int_0^1 ds\frac{\tilde{\alpha}}{(1-s\tilde{\alpha})^{2+\epsilon/2}}+\epsilon\int_0^1 ds\frac{\ln s\tilde{\alpha}}{(1-s\tilde{\alpha})^{1+\epsilon/2}}. \tag{B12}$$

A_{112} should not concern us because it contains terms of at least first order in ϵ. We only calculate A_{111} as

$$A_{111} = \left[\frac{1}{(1+\epsilon/2)(1-s\tilde{\alpha})^{1+\epsilon/2}}\right]_0^1 = \frac{1}{(1+\epsilon/2)}\left[-1+\frac{1}{(1-\tilde{\alpha})^{1+\epsilon/2}}\right]. \tag{B13}$$

Substituting A_{111} into Eq. (B9), we obtain

$$\int_0^1 dz \int_0^1 dt \frac{z^{\epsilon/2}}{(1+\epsilon/2)t^{\epsilon/2}(1-\frac{zt}{4})^{2+\epsilon/2}}\left[\frac{1}{(1-\tilde{\alpha})^{1+\epsilon/2}}-1\right]$$

$$\begin{aligned} &= \int_0^1 dz \int_0^1 dt \frac{z^{\epsilon/2}}{(1+\epsilon/2)t^{\epsilon/2}(1-\frac{zt}{4})^{2+\epsilon/2}[1-\frac{(1-t/2)^2}{(1-zt/4)}]^{1+\epsilon/2}} - \int_0^1 dz \int_0^1 dt \frac{z^{\epsilon/2}}{t^{\epsilon/2}(1+\epsilon/2)(1-\frac{zt}{4})^{2+\epsilon/2}} \\ &= B_{1111} - B_{1112}, \end{aligned} \tag{B14}$$

where we separate the integrand into two parts:

$$\begin{aligned} B_{1111} &= \frac{1}{1+\epsilon/2}\int_0^1\int_0^1 dtdz \frac{z^{\epsilon/2}}{t^{\epsilon/2}(1-\frac{zt}{4})t^{1+\epsilon/2}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}} \\ &= \frac{1}{1+\epsilon/2}\int_0^1 dz \quad z^{\epsilon/2}\int_0^1 dt \frac{1}{(1-\frac{zt}{4})t^{1+\epsilon}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}} . \end{aligned} \tag{B15}$$

The integral over the variable t in Eq. (B15) is given by

$$\begin{aligned} &\int_0^1 dt \frac{z^{\epsilon/2}}{t^{1+\epsilon}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}}\left(\frac{1}{1-\frac{zt}{4}}-1\right) + \int_0^1 dt \frac{z^{\epsilon/2}}{t^{1+\epsilon}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}} \\ &= \int_0^1 dt \frac{\frac{1}{4}z\times z^{1+\epsilon/2}}{t^{\epsilon}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}(1-\frac{zt}{4})} + \int_0^1 dt \frac{z^{\epsilon/2}}{t^{1+\epsilon}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}} \\ &= C_{1111} + C_{1112}. \end{aligned} \tag{B16}$$

The first term C_{1111} is substituted into Eq. (B15), and the contribution is denoted by B_{11111}, which reads

$$\begin{aligned} B_{11111} &= \int_0^1\int_0^1 dtdz \frac{\frac{1}{4}z}{(1-\frac{z}{4}-\frac{t}{4})(1-\frac{zt}{4})} + \cdots \text{(irrelevant terms)} \\ &= 4[9\ln(\tfrac{3}{4}) + 4\ln 2 - \tfrac{1}{2}(\ln 2)^2 + \Phi(1,2) - \Phi(\tfrac{1}{2},2) - \Xi(\tfrac{1}{4},2)] . \end{aligned} \tag{B17}$$

The evaluation of the integral is quite straightforward although tedious. One can find the basic integrals in Appendix C, whose compositions will be used to represent those complex integrals encountered in Eq. (B15).

Here we take the evaluation of B_{11111} as an example and set aside the rest of similar calculations:

$$\begin{aligned} B_{11111} &= \int_0^1\int_0^1 dzdt\,\frac{1}{4}z\left[\frac{1}{(1-\frac{z}{4}-\frac{t}{4})}-\frac{t}{(1-\frac{zt}{4})}\right]\frac{1}{(1-t+\frac{t^2}{4})} \\ &= \int_0^1\int_0^1 dzdt\,\frac{1}{\tilde{\Delta}}\left[\frac{(1-\frac{t}{4})}{(1-\frac{z}{4}-\frac{t}{4})}-\frac{1}{(1-\frac{zt}{4})}\right] \\ &= -4\int_0^1 dt\left[\frac{(1-\frac{t}{4})}{(1-\frac{t}{2})^2}\ln\left(\frac{3}{4}-\frac{t}{4}\right)-\frac{\ln(1-\frac{t}{4})}{1-\frac{t}{2})^2}\left(1-\frac{t}{4}\right)-\frac{\ln(1-\frac{t}{4})}{(1-\frac{t}{2})^2 t}\right], \end{aligned} \tag{B18}$$

where $\tilde{\Delta} = 1 - t - \frac{t^2}{4}$. Let $u = 1 - \frac{t}{2}$. Equation (B18) turns into

$$B_{11111} = -2\times 1/2\left[\int_1^{1/2} du \frac{\ln(\frac{1}{4}+\frac{u}{2})}{u^2} + \frac{\ln(\frac{1}{4}+\frac{u}{2})}{u} - \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u^2} - \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u} - \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u^2} - \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u} + \frac{\ln(\frac{1}{2}+\frac{u}{2})}{1-u}\right]. \tag{B19}$$

Each term in Eq. (B18) can be easily represented in terms of the basic integrals listed in Appendix C. For example,

$$\int_1^{1/2} \frac{\ln(\frac{1}{4}+\frac{u}{2})}{u^2} = 2\ln(2) + 3\ln(\tfrac{3}{4}) \ , \qquad \text{(B20)}$$

$$\int_1^{1/2} du \frac{\ln(\frac{1}{4}+\frac{u}{2})}{u} = [2\ln^2(2) - \tfrac{1}{2}\ln^2(2) + \Phi(\tfrac{1}{2},2) - \Phi(1,2)] \ , \qquad \text{(B21)}$$

$$\int_1^{1/2} du \left[\frac{\ln(1+u)}{u^2} - \frac{\ln(2)}{u^2}\right] = -3\ln(\tfrac{3}{2}) + 2\ln(2) \ , \qquad \text{(B22)}$$

$$\int_1^{1/2} du \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u} = \ln^2(2) + \Phi(\tfrac{1}{2},2) - \Phi(1,2) \ , \qquad \text{(B23)}$$

$$\int_1^{1/2} du \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u^2} = 2\ln(2) - 3\ln(\tfrac{3}{2}) \ , \qquad \text{(B24)}$$

$$\int_1^{1/2} du \frac{\ln(\frac{1}{2}+\frac{u}{2})}{(1-u)} = \Xi(\tfrac{1}{4},2) \ , \qquad \text{(B25)}$$

$$\int_1^{1/2} du \frac{(1+2u)}{u} = -\tfrac{1}{2}\ln^2(2) - \Phi(1,2) + \Phi(\tfrac{1}{2},2) \ , \qquad \text{(B26)}$$

$$\int_1^{1/2} du \frac{(1+u)}{u} = \Phi(\tfrac{1}{2},2) - \Phi(1,2), \qquad \text{(B27)}$$

where Φ and Ξ are defined in Appendix C.

Now we turn to the calculation of B_{11112}, which is defined as $B_{11112} = \int_0^1 dt C_{1112}$:

$$B_{11112} = \int_0^1 \int_0^1 dz dt \frac{z^{\epsilon/2}}{t^{1+\epsilon}(1-\frac{z}{4}-\frac{t}{4})^{1+\epsilon/2}} = \int_0^1 dz \frac{z^{\epsilon/2}}{(1-\frac{z}{4})^{1+\epsilon/2}} \times \int_0^1 dt \frac{1}{t^{1+\epsilon}\left[1-\frac{t}{4(1-z/4)}\right]^{1+\epsilon/2}}, \qquad \text{(B28)}$$

where the t-dependent integral can be separated into

$$\int_0^1 dt \frac{1}{t^{1+\epsilon}(1-\tilde{\beta}t)^{1+\epsilon/2}} = \int_0^1 dt \frac{1}{t^{1+\epsilon}} \frac{[1-(1-\tilde{\beta}t)^{1+\epsilon/2}]}{(1-\tilde{\beta})^{1+\epsilon/2}} + \int_0^1 dt \frac{1}{t^{1+\epsilon}} = \int_0^1 dt \frac{1}{t^{\epsilon}} \frac{\tilde{\beta}}{(1-\tilde{\beta})^{1+\epsilon/2}} + O(\epsilon) + \int_0^1 dt \frac{1}{t^{1+\epsilon}} = D_1 + D_2 + O(\epsilon) \ , \qquad \text{(B29)}$$

with $\tilde{\beta}$ being $\frac{1}{4-z}$. The term D_1 in Eq. (B29) is

$$D_1 = \int_0^1 dt \frac{\tilde{\beta}}{(1-\tilde{\beta}t)} = -\ln(1-\tilde{\beta}). \qquad \text{(B30)}$$

After substituting the above equation into Eq. (B28), one has

$$\int_0^1 dz(-1) \frac{z^{\epsilon/2}\ln(1-\frac{1}{4-z})}{(1-\frac{z}{4})^{1+\epsilon/2}} = \int_0^1 dz \frac{[\ln(4-z)-\ln(3-z)]}{(1-\frac{z}{4})} + O(\epsilon) = I_A - I_B + 4\ln^2(\tfrac{4}{3}). \qquad \text{(B31)}$$

Inserting D_2 in Eq. (B29) into Eq. (B28), we have

$$\int_0^1 dt \frac{1}{t^{1+\epsilon}} \times \int_0^1 dz \frac{z^{\epsilon/2}}{(1-\frac{z}{4})^{1+\epsilon/2}} = \frac{-1}{\epsilon} \times \left[\int_0^1 dz \frac{1}{(1-\frac{z}{4})} - \frac{\epsilon}{2}\int_0^1 dz \frac{\ln(1-\frac{z}{4})}{(1-\frac{z}{4})} + \int_0^1 dz \frac{\ln(z)}{(1-\frac{z}{4})}\right] + O(\epsilon) = \left(-\frac{1}{\epsilon}\right)\left[4\ln(\tfrac{4}{3}) - \frac{\epsilon}{2}I_A + \frac{\epsilon}{2}I_C\right] \ . \qquad \text{(B32)}$$

Now we move on to calculate the contribution of A_2 in Eq. (B10). By inserting A_2 into Eq. (B9), one has

$$\int_0^1\int_0^1\int_0^1 dz dt ds \frac{z^{\epsilon/2}}{t^{\epsilon/2}} \frac{1}{s^{1+\epsilon}(1-\frac{zt}{4})^{2+\epsilon/2}} = \int_0^1\int_0^1\int_0^1 dz ds dt \frac{1}{s^{1+\epsilon}(1-\frac{zt}{4})^2} + \int_0^1\int_0^1\int_0^1 dz dt ds \frac{1}{s^{1+\epsilon}(1-\frac{zt}{4})^2} \times \left[\left(-\frac{\epsilon}{2}\right)\ln\left(1-\frac{zt}{4}\right) + \frac{\epsilon}{2}\ln(z) - \frac{\epsilon}{2}\ln(t)\right] + \cdots = 4\ln(\tfrac{4}{3})\frac{1}{-\epsilon} + \int_0^1\int_0^1 dz dt \left(\frac{1}{-\epsilon}\right)\left(\frac{\epsilon}{2}\right)\left[\frac{-\ln(1-\frac{zt}{4}) + \ln(z) - \ln(t)}{(1-\frac{zt}{4})^2}\right]. \qquad \text{(B33)}$$

Here we decompose the second term in Eq. (B33) into three parts, P_1 , P_2, and P_3:

$$P_1 = \int_0^1\int_0^1 dz dt \frac{\ln(1-\frac{zt}{4})}{(1-\frac{zt}{4})^2} = \int_0^1 dt \left(\frac{-4}{t}\right) \frac{\ln(1-\frac{zt}{4})}{(1-\frac{zt}{4})}\Big|_0^1 + \int_0^1\int_0^1 dz dt \frac{(-1)}{(1-\frac{zt}{4})^2} = \int_0^1 dt \quad \frac{(-4)}{t}\frac{\ln(1-\frac{t}{4})}{(1-\frac{t}{4})} + [-4\ln(\tfrac{4}{3})] = -\int_0^1 dt \quad 4\frac{\ln(1-\frac{t}{4})}{t} - \int_0^1 dt \frac{\ln(1-\frac{t}{4})}{(1-\frac{t}{4})} - 4\ln(\tfrac{4}{3}) = -[4I_D + I_A] - 4\ln(\tfrac{4}{3}) \ , \qquad \text{(B34)}$$

$$P_2 = \int_0^1 \int_0^1 dtdz \frac{\ln(z)}{(1-\frac{zt}{4})^2} = \int_0^1 \frac{\ln(z)}{(1-\frac{z}{4})} = I_C \ , \tag{B35}$$

$$P_3 = \int_0^1 \int_0^1 dtdz \frac{\ln(t)}{(1-\frac{zt}{4})^2} = I_C. \tag{B36}$$

Now we go back to finish the calculation of A_{12} in Eq. (B11):

$$\begin{aligned} A_{12} &= \int_0^1 \int_0^1 dzdt \frac{\ln\left(1 - \frac{(1-t/2)^2}{1-\frac{zt}{4}}\right)}{(1-\frac{zt}{4})^2} \\ &= -\int_0^1 \int_0^1 dzdt \frac{\ln(1-\frac{zt}{4})}{(1-\frac{zt}{4})^2} + \int_0^1 \int_0^1 dzdt \frac{\ln(t) + \ln(1-\frac{z}{4}-\frac{t}{4})}{(1-\frac{zt}{4})^2} \\ &= [4I_D + I_A + 4\ln(\tfrac{4}{3})] + I_C + \int_0^1 dt \frac{4}{t} \frac{\ln(1-\frac{z}{t}-\frac{t}{4})}{(1-\frac{zt}{4})} \Big|_0^1 - \int_0^1 \int_0^1 dzdt \frac{4\times(-\frac{1}{4})}{t(1-\frac{zt}{4})(1-\frac{z}{4}-\frac{t}{4})} \\ &= [4I_D + I_A + 4\ln(\tfrac{4}{3})] + I_C + L_1 - M_1 + M_2, \end{aligned} \tag{B37}$$

where L_1, M_1, and M_2 are analyzed below:

$$\begin{aligned} L_1 &= \int_0^1 dt \frac{4}{t} \left[\frac{\ln(\frac{3}{4}-\frac{t}{4})}{(1-\frac{t}{4})} - \ln\left(1-\frac{t}{4}\right) \right] \\ &= \int_0^1 dt \left\{ \left[\frac{4}{t} + \frac{1}{(1-\frac{t}{4})} \right] \ln\left(\frac{3}{4}-\frac{t}{4}\right) - \frac{4}{t} \ln\left(1-\frac{t}{4}\right) \right\} \\ &= \int_0^1 dt \frac{4}{t} \ln\left(\frac{3}{4}-\frac{t}{4}\right) + \left[-4\ln\left(\frac{3}{4}\right) + I_B - 4I_D \right]. \end{aligned} \tag{B38}$$

The first term in Eq. (B38) is divergent. As we will show later, it will be canceled by a term in M_2.

$$\begin{aligned} M_1 &= 8\int_0^1 dt \frac{1}{t} \left(\frac{-1}{\Delta} + 1 \right) \ln\left(1-\frac{t}{4}\right) - 8\int_0^1 dt \frac{1}{t} \ln\left(1-\frac{t}{4}\right) \\ &= 8\int_1^{1/2} du \left[\frac{\ln(\frac{1}{2}+\frac{u}{2})}{u^2} + \frac{\ln(\frac{1}{2}+\frac{u}{2})}{u} \right] - 8\int_0^1 dt \frac{1}{t} \ln\left(1-\frac{t}{4}\right) \\ &= 8[2\ln(2) - 3\ln(\tfrac{3}{2}) + \ln^2(2) + \Phi(\tfrac{1}{2},2) - \Phi(1,2)] - 8I_D, \end{aligned} \tag{B39}$$

where $\Delta = (1-\frac{t}{2})^2$ and $u = 1-\frac{t}{2}$.

$$\begin{aligned} M_2 &= \int_0^1 dt \frac{4}{t} \ln\left(\frac{3}{4}-\frac{1}{4}t\right)\left(\frac{-1}{\Delta}+1\right) - \int_0^1 dt \quad \frac{4}{t} \quad \ln\left(\frac{3}{4}-\frac{t}{4}\right) \\ &= 4\int_1^{1/2} du \left(\frac{1+u}{u^2}\right) \ln\left(\frac{1}{4}+\frac{u}{2}\right) - \int_0^1 dt \frac{4}{t} \ln\left(\frac{3}{4}-\frac{t}{4}\right) \\ &= 4[2\ln(2) + 3\ln(\tfrac{3}{4}) + 2\ln^2(2) - \tfrac{1}{2}\ln^2(2) + \Phi(\tfrac{1}{2},2) - \Phi(1,2)] - \int_0^1 dt \frac{4}{t} \ln\left(\frac{3}{4}-\frac{t}{4}\right), \end{aligned} \tag{B40}$$

where, as mentioned above, the first term in Eq. (B38) is canceled by the last term in Eq. (B40).

The term B_{1112} in Eq. (B14) is calculated below:

$$B_{1112} = \frac{1}{1+\epsilon/2} \int_0^1 \int_0^1 dzdt \frac{z^{\epsilon/2}}{t^{\epsilon/2}(1-\frac{zt}{4})^{2+\epsilon/2}} = \int_0^1 \int_0^1 dzdt \frac{1}{(1-\frac{zt}{4})^2} + O(\epsilon) = 4\ln(\tfrac{4}{3}) + O(\epsilon). \tag{B41}$$

The second term $I_{3.1-2}$ in Eq. (B3) is calculated as follows:

$$\begin{aligned} I_{3.1-2} = &\int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)[(\vec{p}-\vec{k})^2+m^2]\{(k^2+m^2)+[k^2+(\vec{p}-\vec{k})^2+2m^2]\}} \\ &\times \frac{1}{\{[q^2+(\vec{p}-\vec{q})^2+2m^2]+[k^2+(\vec{p}-\vec{k})^2+2m^2]\}} \\ = &\frac{\Gamma(1+1+1)}{4[\Gamma(1)]^3} \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)} \int_0^1 \int_{0\ x+y<1}^1 dxdy \end{aligned}$$

$$\times\{(1-x-y)[(\vec{p}-\vec{k})^2+m^2]+x(k^2+p^2-\vec{k}\cdot\vec{p}+m^2)$$
$$+y(p^2+q^2+k^2-\vec{p}\cdot\vec{q}-\vec{p}\cdot\vec{k}+m^2)\}^{-3}+\text{(irrelevant terms)}$$
$$=\frac{\Gamma(3)}{4}\int_{-\infty}^{\infty}d^d\vec{p}\int_{-\infty}^{\infty}d^d\vec{q}\frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)}\int_0^1\int_{0\ x+y<1}^1 dxdy\int_0^1 dz(\pi)^{d/2}$$
$$\times\frac{\Gamma(3-1-\epsilon/2)}{\Gamma(3)}\frac{1}{\{[1-(1-\frac{x+y}{2})^2]p^2+\vec{p}\cdot\vec{q}(-y)+yq^2+m^2\}^{2-\epsilon/2}}$$
$$=(\pi)^{d/2}\frac{\Gamma(2-\epsilon/2)}{4}\int_{-\infty}^{\infty}d^d\vec{q}\int_0^1\int_{0\ x+y<1}^1 dxdy\frac{1}{(q^2+m^2)}(\pi)^{d/2}\frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon/2)}$$
$$\times\frac{\vec{q}\cdot(\frac{y}{2\Delta}\vec{q})}{(\Delta)^{2-\epsilon/2}\{[\frac{y}{\Delta}-(\frac{y}{2\Delta})^2]q^2+\frac{m^2}{\Delta}\}^{1-\epsilon}}$$
$$=\pi^d\frac{\Gamma(1-\epsilon)}{8}\int_0^1\int_{0\ x+y<1}^1 dxdy\frac{y}{\Delta^{3-\epsilon/2}\Lambda^{1-\epsilon}}\left[\frac{m}{(\Lambda\Delta)^{1/2}}\right]^{2+\epsilon-2(1-\epsilon)}\pi^{d/2}\frac{\Gamma(-\frac{3}{2}\epsilon)}{\Gamma(1-\epsilon)}$$
$$=\frac{\pi^{3d/2}\Gamma(-\frac{3}{2}\epsilon)}{8}(m^2)^{3/2\epsilon}\int_0^1\int_{0\ x+y<1}^1 dxdy\frac{y}{\Delta^{3+\epsilon}(\frac{y}{\Delta}-\frac{y^2}{4\Delta^2})^{1+\epsilon/2}},\qquad\text{(B42)}$$

where $\Delta=(x+y)^2-\frac{1}{4}(x+y)^2, \Lambda=\frac{y}{\Delta}-\frac{y}{2\Delta}$. The integral in Eq. (B42) is analyzed below:

$$\int_0^1\int_{0\ x+y<1}^1 dxdy\frac{y}{\Delta^{3+\epsilon}}\frac{1}{[\frac{y^{1+\epsilon/2}}{\Delta^{2+\epsilon}}](\Delta-\frac{y}{4})^{1+\epsilon/2}}$$
$$=\int_0^1\int_0^1 dsdt\frac{s\times st}{s(1-\frac{s}{4})s^{1+\epsilon/2}t^{1+\epsilon/2}}\frac{1}{(s-\frac{s^2}{4}-\frac{st}{4})^{1+\epsilon/2}}$$
$$=\int_0^1\int_0^1 dsdt\frac{1}{s^{1+\epsilon}t^{\epsilon/2}(1-\frac{s}{4}-\frac{t}{4})^{1+\epsilon/2}}\left[\frac{1}{1-\frac{s}{4}}-1\right]+\int_0^1\int_0^1 dsdt\frac{1}{s^{1+\epsilon}t^{\epsilon/2}}\frac{1}{(1-\frac{s}{4}-\frac{t}{4})^{1+\epsilon/2}}$$
$$=S+R_1+R_2+O(\epsilon),\qquad\text{(B43)}$$

where the transformations of $y=st$ and $x=s(1-t)$ have been used, and the calculations for S, R_1, and R_2 will be presented successively:

$$S=\int_0^1\int_0^1 dsdt\frac{\frac{1}{4}}{(1-\frac{s}{4}-\frac{t}{4})(1-\frac{s}{4})}=\int_0^1 ds\frac{1}{1-\frac{s}{4}}\left[-\ln\left(\frac{3}{4}-\frac{s}{4}\right)+\ln\left(1-\frac{s}{4}\right)\right]=[I_A+4\ln^2(\tfrac{4}{3})-I_B]\ .\qquad\text{(B44)}$$

The reason for the rise of R_1 and R_2 can be revealed by the following decomposition:

$$R_1+R_2=\int_0^1\int_0^1 dsdt\frac{1}{s^{1+\epsilon}t^{\epsilon/2}[1-\frac{s}{4}-\frac{t}{4}]^{1+\epsilon/2}}$$
$$=\int_0^1\int_0^1 dsdt\frac{1}{t^{\epsilon/2}(1-\frac{t}{4})^{1+\epsilon/2}}\frac{1}{s^{1+\epsilon}(1-\frac{s}{4-t})^{1+\epsilon/2}}$$
$$=\int_0^1 dt\frac{1}{t^{\epsilon/2}(1-\frac{t}{4})^{1+\epsilon/2}}\int_0^1 ds\frac{1}{s^{1+\epsilon}}\left[\frac{1}{(1-\frac{s}{4-t})^{1+\epsilon/2}}-1\right]+\int_0^1 dt\frac{1}{t^{\epsilon/2}(1-\frac{t}{4})^{1+\epsilon/2}}\int_0^1 ds\frac{1}{s^{1+\epsilon}},\qquad\text{(B45)}$$

where

$$R_1=\int_0^1 dt\frac{1}{t^{\epsilon/2}(1-\frac{t}{4})^{1+\epsilon/2}}\int_0^1 ds\frac{1}{s^{\epsilon}(4-t)(1-\frac{s}{4-t})^{1+\epsilon/2}}$$
$$=\int_0^1 dt\frac{1}{(1-\frac{t}{4})(4-t)}\int_0^1 ds\frac{1}{(1-\frac{s}{4-t})}+O(\epsilon)$$
$$=\int_0^1 dt\frac{[\ln(4-t)-\ln(3-t)]}{(1-\frac{t}{4})}=4\ln^2(\tfrac{4}{3})+I_A-I_B,\qquad\text{(B46)}$$

and

$$
\begin{aligned}
R_2 &= \int_0^1 dt \frac{1}{t^\epsilon (1-\frac{t}{4})^{1+\epsilon/2}} \int_0^1 ds \frac{1}{s^{1+\epsilon}} \\
&= \int_0^1 dt \frac{1}{1-\frac{t}{4}} \left(\frac{1}{-\epsilon}\right) + \int_0^1 dt \frac{[\ln(t) + 1/2 \ln(1-\frac{t}{4})]}{(1-\frac{t}{4})} + O(\epsilon) \\
&= 4 \ln\left(\frac{4}{3}\right) \left(\frac{1}{-\epsilon}\right) + I_C + \frac{1}{2} I_A.
\end{aligned} \tag{B47}
$$

The term $I_{3.1-3}$ will be equal to $I_{3.2-2}$, which will be shown later.

For the other three-loop diagram, we can still employ the same technique. To begin with, we write down the corresponding integral for Fig. 15:

$$
\begin{aligned}
I_{\text{FD }312} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{(-\vec{p}\cdot\vec{k}) \times (-\vec{p}\cdot\vec{q})}{(k^2+m^2)(p^2+m^2)(q^2+m^2)} \int_0^\infty dt_x \int_{-\infty}^{t_x} dt_y \\
\times e^{-[k^2+m^2+(\vec{p}-\vec{k})^2+m^2](t_x-t_y)} e^{-[q^2+m^2+(\vec{p}-\vec{q})^2+m^2]t_x} e^{-(p^2+m^2)|t_y|}.
\end{aligned} \tag{B48}
$$

Again, the time dependent part of the integral in Eq. (B48) can be integrated first. For convenience, let $a = (p^2+m^2), b = [(\vec{p}-\vec{q})^2+m^2], c = (k^2+m^2), d = [(\vec{p}-\vec{k})^2+m^2]$, and $e = (p^2+m^2)$. Then we have

$$
\begin{aligned}
&\int_0^\infty dt_x e^{-[c+d+a+b]t_x} \left[\int_0^{t_x} dt_y e^{(c+d-e)t_y} + \int_{-\infty}^0 dt_y e^{(c+d+e)t_y}\right] \\
&= \int_0^\infty dt_x e^{-(a+b+c+d)t_x} \left[\frac{1}{c+d-e}(e^{(c+d-e)t_x} - 1) + \frac{1}{c+d+e}\right] \\
&= \frac{2}{(c+d+e)(a+b+c+d)}.
\end{aligned} \tag{B49}
$$

In the same spirit of the calculation as performed in the previous case, we separate the integrals into several pieces, each of which only contains an isolated pole, and then extract the corresponding singular parts. One can start with decomposing $\vec{p}\cdot\vec{k}$ into:

$$
\vec{p}\cdot\vec{k} = \frac{1}{2}\{-[p^2+k^2+(\vec{p}-\vec{k})^2+3m^2] + 2(p^2+m^2) + 2(k^2+m^2)\}. \tag{B50}
$$

Then one can rewrite Eq. (B48) as $-I_{3.2-1} + 2I_{3.2-2} + 2I_{3.2-3}$, where

$$
I_{3.2-1} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(k^2+m^2)(q^2+m^2)(p^2+m^2)} \frac{1}{[k^2+q^2+(\vec{p}-\vec{q})^2+(\vec{p}-\vec{k})^2+4m^2]}, \tag{B51}
$$

$$
\begin{aligned}
I_{3.2-2} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(k^2+m^2)(q^2+m^2)[p^2+k^2+(\vec{p}-\vec{k})^2+3m^2]} \\
\times \frac{1}{[k^2+q^2+(\vec{p}-\vec{q})^2+(\vec{p}-\vec{k})^2+4m^2]},
\end{aligned} \tag{B52}
$$

$$
\begin{aligned}
I_{3.2-3} = \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)[p^2+k^2+(\vec{p}-\vec{k})^2+3m^2]} \\
\times \frac{1}{[k^2+q^2+(\vec{p}-\vec{q})^2+(\vec{p}-\vec{k})^2+4m^2]}.
\end{aligned} \tag{B53}
$$

Again, we have three different types of the integrals to handle.

The calculation for the term $I_{3.2-1}$ is shown below:

$$
\begin{aligned}
I_{3.2-1} &= \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(k^2+m^2)(q^2+m^2)(p^2+m^2)} \frac{1}{k^2+q^2+(\vec{q}-\vec{p})^2+(\vec{p}-\vec{k})^2+4m^2} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)} \int_0^1 dx \int_{-\infty}^{\infty} d^d\vec{k} \frac{1}{[k^2 - x\vec{p}\cdot\vec{k} + xp^2 + xq^2 - x\vec{p}\cdot\vec{q} + m^2]^2} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} d^d\vec{q} \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)} \int_0^1 dx \pi^{d/2} \frac{\Gamma(1-\epsilon/2)}{\Gamma(2)} \frac{1}{[(x-\frac{x^2}{4})p^2 - x\vec{p}\cdot\vec{q} + xq^2 + m^2]^{1-\epsilon/2}}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\pi^{d/2}\Gamma(1-\epsilon/2)}{2}\int_{-\infty}^{\infty} d^d\vec{p}\int_{-\infty}^{\infty} d^d\vec{q}\frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)}\frac{1}{(x-\frac{x^2}{4})^{1-\epsilon/2}(p^2-\frac{x}{\Delta}\vec{p}\cdot\vec{q}+\frac{x}{\Delta}q^2+\frac{m^2}{\Delta})^{1-\epsilon/2}} \\
&= \frac{\pi^{d/2}\Gamma(1-\epsilon/2)}{2}\frac{\Gamma(2-\epsilon/2)}{\Gamma(1-\epsilon/2)}\int_{-\infty}^{\infty} d^d\vec{p}\int_{-\infty}^{\infty} d^d\vec{q}\int_0^1\int_0^1 dxdy\frac{1}{(q^2+m^2)\Delta^{1-\epsilon/2}} \\
&\quad\times\frac{\vec{p}\cdot\vec{q}y^{-\epsilon/2}}{[p^2-\frac{xy}{\Delta}\vec{p}\cdot\vec{q}+\frac{xy}{\Delta}q^2+\frac{y}{\Delta}m^2]^{2-\epsilon/2}} \\
&= \frac{\pi^{d}\Gamma(1-\epsilon)}{2}\int_0^1\int_0^1 dxdy\int_{-\infty}^{\infty} d^d\vec{q}\frac{y^{-\epsilon/2}}{(q^2+m^2)\Delta^{1-\epsilon/2}}\frac{\vec{q}\cdot(\frac{xy}{2\Delta}\vec{q})}{\{[\frac{xy}{\Delta}-(\frac{xy}{2\Delta})^2]q^2+\frac{y}{\Delta}m^2\}^{1-\epsilon}} \\
&= \frac{\pi^{d}\Gamma(1-\epsilon)}{2}\int_0^1\int_0^1 dxdy\int_{-\infty}^{\infty} d^d\vec{q}\frac{(xy)y^{-\epsilon/2}}{2\Delta\Lambda^{1-\epsilon}\Delta^{1-\epsilon/2}}\frac{1}{(q^2+\frac{y}{\Lambda\Delta}m^2)^{1-\epsilon}} \\
&= \frac{\pi^{3d/2}}{4}\Gamma\left(-\frac{3\epsilon}{2}\right)(m^2)^{3\epsilon/2}\int_0^1\int_0^1 dxdy\frac{xy^{1+3\epsilon/2}y^{-\epsilon/2}}{\Lambda^{1+\epsilon/2}\Delta^{2+\epsilon}},
\end{aligned}
\tag{B54}
$$

where $\Delta = x-\frac{x^2}{4}$ and $\Lambda = \frac{xy}{\Delta}-(\frac{xy}{2\Delta})^2$. The integration over x and y in Eq. (B54) is carried out as

$$
\begin{aligned}
\int_0^1\int_0^1 dxdy\frac{xy^{1+\epsilon}}{(xy)^{1+\epsilon/2}(\Delta-\frac{xy}{4})^{1+\epsilon/2}} &= \int_0^1\int_0^1 dxdy\frac{y^{\epsilon/2}}{x^{1+\epsilon}(1-\frac{x}{4}-\frac{y}{4})^{1+\epsilon/2}} \\
&= \int_0^1 dy\frac{y^{\epsilon/2}}{(1-\frac{y}{4})^{1+\epsilon/2}}\int_0^1 dx\frac{1}{x^{1+\epsilon}(1-\frac{x}{4-y})^{1+\epsilon/2}}.
\end{aligned}
\tag{B55}
$$

Rewrite Eq. (B55) as

$$
\begin{aligned}
&\int_0^1 dy\frac{y^{\epsilon/2}}{(1-\frac{y}{4})^{1+\epsilon/2}}\left\{\int_0^1 dx\frac{1}{x^{1+\epsilon}}\left[\frac{1}{(1-\frac{x}{4-y})^{1+\epsilon/2}}-1\right]+\int_0^1\frac{1}{x^{1+\epsilon}}\right\} \\
&\qquad = \int_0^1 dy\frac{y^{\epsilon/2}}{(1-\frac{y}{4})^{1+\epsilon/2}}\left[-\ln\left(1-\frac{x}{4-y}\right)\Bigg|_0^1+\frac{1}{-\epsilon}+O(\epsilon)\right].
\end{aligned}
\tag{B56}
$$

One also expands the terms in Eq. (B55) up to the zero order in ϵ. The finite part of the first term in Eq. (B56) is found to be

$$
\int_0^1 dy\frac{1}{(1-\frac{y}{4})}[\ln(4-y)-\ln(3-y)] = 4(\ln\tfrac{4}{3})^2+I_A-I_B.
\tag{B57}
$$

The second part is extracted in the same manner:

$$
\begin{aligned}
\int_0^1 dy\frac{y^{\epsilon/2}}{(1-\frac{x}{4-y})^{1+\epsilon/2}} &= \left\{\int_0^1 dy\frac{1}{(1-\frac{y}{4})}+\frac{\epsilon}{2}\left[\frac{\ln(y)}{(1-\frac{y}{4})}-\frac{\ln(1-\frac{y}{4})}{1-\frac{y}{4}}\right]\right\}+O(\epsilon) \\
&= \left[4\ln\left(\frac{4}{3}\right)+\frac{\epsilon}{2}(I_C-I_A)\right].
\end{aligned}
\tag{B58}
$$

Now we turn to the term $I_{3.2-2}$:

$$
\begin{aligned}
I_{3.2-2} &= \int_{-\infty}^{\infty} d^d\vec{p}\int_{-\infty}^{\infty} d^d\vec{q}\int_{-\infty}^{\infty} d^d\vec{k}\int_0^1\int_{0\ x+y<1}^1 dxdy\frac{1}{4(q^2+m^2)}\frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)} \\
&\quad\times\frac{1}{[k^2-(x+y)\vec{p}\cdot\vec{k}+(x+y)p^2+yq^2-y\vec{p}\cdot\vec{q}+m^2]^3} \\
&= \frac{\Gamma(3)}{\Gamma(1)^3}\int_{-\infty}^{\infty} d^d\vec{p}\int_{-\infty}^{\infty} d^d\vec{q}\frac{1}{4(q^2+m^2)}\int_0^1\int_{0\ x+y<1}^1 dxdy\pi^{d/2}\frac{\Gamma(2-\epsilon/2)}{\Gamma(3)} \\
&\quad\times\frac{\vec{p}\cdot\vec{q}}{\Delta^{2-\epsilon/2}[p^2+\frac{y}{\Delta}q^2-\frac{y}{\Delta}\vec{p}\cdot\vec{q}+\frac{m^2}{\Delta}]^{2-\epsilon/2}} \\
&= \int_{-\infty}^{\infty} d^d\vec{q}\frac{1}{4(q^2+m^2)}\int_0^1\int_{0\ x+y<1}^1 dxdy\pi^{d}\frac{\Gamma(2-\epsilon/2)}{\Gamma(3)}\frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon/2)}\frac{\vec{q}\cdot(\frac{y}{2\Delta})\vec{q}}{\Delta^{2-\epsilon/2}\{[\frac{y}{\Delta}-(\frac{y}{2\Delta})^2]q^2+\frac{m^2}{\Delta}\}^{1-\epsilon}}
\end{aligned}
$$

$$= \pi^d \frac{\Gamma(1-\epsilon)}{8} \int_{-\infty}^{\infty} d^d\vec{q} \frac{y}{\Delta^{3-\epsilon/2}\Lambda^{1-\epsilon}} \frac{q^2}{(q^2+\frac{m^2}{\Lambda\Delta})^{1-\epsilon}(q^2+m^2)} (\pi)^{d/2} \frac{\Gamma(2+\epsilon/2)}{\Gamma(1+\epsilon/2)} \frac{\Gamma(-3\epsilon/2)}{\Gamma(1-\epsilon)}$$

$$= \pi^{3d/2}\Gamma(-3\epsilon/2) \int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{1}{8} \frac{\Gamma(2+\epsilon/2)}{\Gamma(1+\epsilon/2)} \frac{y}{\Delta^{3+\epsilon}\Lambda^{1+\epsilon/2}}. \tag{B59}$$

Here $\Delta = (x+y) - \frac{1}{4}(x+y)^2$ and $\Lambda = \frac{y}{\Delta} - (\frac{y}{2\Delta})^2$. The integration over x and y in Eq. (B59) is performed as

$$\int_0^1 \int_{0\ x+y<1}^1 dxdy \frac{y}{[\frac{y}{\Delta} - (\frac{y}{2\Delta})^2]^{1+\epsilon/2}\Delta^{3+\epsilon}} = \int_0^1 \int_0^1 dsdt \frac{s^2 t}{(st)^{1+\epsilon/2}\Delta(\Delta - \frac{st}{4})^{1+\epsilon/2}}$$

$$= \int_0^1 \int_0^1 dsdt \frac{1}{s^{1+\epsilon}t^{\epsilon/2}(1-\frac{s}{4})[1-\frac{s}{4}-\frac{t}{4}]^{1+\epsilon/2}}. \tag{B60}$$

The s dependent part in Eq. (B60) can be separated into

$$\int_0^1 ds \frac{1}{s^{1+\epsilon}(1-\frac{s}{4-t})^{1+\epsilon/2}} \left[\frac{1}{1-\frac{s}{4}} - 1\right] + \int_0^1 ds \frac{1}{s^{1+\epsilon}(1-\frac{s}{4-t})^{1+\epsilon/2}}$$

$$= \int_0^1 ds \frac{\frac{1}{4}}{(1-\frac{s}{4-t})(1-\frac{s}{4})} + \int_0^1 ds \frac{\frac{1}{4-t}}{s^{\epsilon}(1-\frac{s}{4-t})^{1+\epsilon/2}} + \int_0^1 ds \frac{1}{s^{1+\epsilon}} + O(\epsilon). \tag{B61}$$

The first term in Eq. (B61) can be integrated out as

$$\int_0^1 ds \frac{\frac{1}{4}}{(1-\frac{s}{4-t})(1-\frac{s}{4})} = \left[\ln\left(1-\frac{t}{4}\right) - \ln\left(1-\frac{t}{3}\right)\right]. \tag{B62}$$

Inserting it into Eq. (B60), we inherit

$$\int_0^1 dt \left(\frac{4-t}{t}\right) \left[\ln\left(1-\frac{t}{4}\right) - \ln\left(1-\frac{t}{3}\right)\right] = 4(I_D - I_E) + 2\ln(\tfrac{3}{2}) - 3\ln(\tfrac{4}{3}). \tag{B63}$$

Again, inserting the second term in Eq. (B61) into Eq. (B60), one has

$$\int_0^1 dt \frac{1}{1-\frac{t}{4}} \left[-\ln\left(1-\frac{1}{4-t}\right)\right] = I_A - I_B + 4\ln^2(\tfrac{4}{3}). \tag{B64}$$

After substituting the third term in Eq. (B61) into Eq. (B60), one obtains

$$\int_0^1 ds \frac{1}{s^{1+\epsilon}} \cdot \int_0^1 dt \left[\frac{1}{1-\frac{t}{4}} - \frac{\epsilon}{2} \frac{[\ln(t) + \ln(1-\frac{t}{4})]}{1-\frac{t}{4}}\right] = \left(\frac{1}{-\epsilon}\right) \left[4\ln\left(\frac{4}{3}\right) + \frac{\epsilon}{2}(-I_C - I_A)\right]. \tag{B65}$$

Here we evaluate the integral of $I_{3.2-3}$:

$$I_{3.2-3} = \frac{1}{4} \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \int_{-\infty}^{\infty} d^d\vec{k} \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)} \int_0^1 dx \frac{1}{(k^2 - \vec{p}\cdot\vec{k} + p^2 + xq^2 - x\vec{p}\cdot\vec{q} + m^2)^2}$$

$$= \frac{1}{4} \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)} \int_0^1 dx \pi^{d/2} \frac{\Gamma(1-\epsilon/2)}{\Gamma(2)} \frac{1}{(\frac{3}{4}p^2 - x\vec{p}\cdot\vec{q} + xq^2 + m^2)^{1-\epsilon/2}}$$

$$= \frac{1}{4} \quad \pi^{d/2} \frac{\Gamma(1-\epsilon/2)}{(\frac{3}{4})^{1-\epsilon/2}} \int_0^1 dx \int_{-\infty}^{\infty} d^d\vec{p} \int_{-\infty}^{\infty} d^d\vec{q} \frac{\vec{p}\cdot\vec{q}}{(p^2+m^2)(q^2+m^2)} \frac{1}{(\frac{3}{4})^{1-\epsilon/2}(p^2 - \frac{4}{3}x\vec{p}\cdot\vec{q} + \frac{4}{3}q^2 + \frac{4}{3}m^2)^{1-\epsilon/2}}$$

$$= \frac{1}{4}\pi^{d/2} \frac{\Gamma(2-\epsilon/2)}{(\frac{3}{4})^{1-\epsilon/2}} \int_0^1 \int_0^1 dxdy \frac{\vec{p}\cdot\vec{q}}{(q^2+m^2)} \frac{y^{-\epsilon/2}}{(p^2 - \frac{4}{3}xy\vec{p}\cdot\vec{q} + \frac{4}{3}xyq^2 + \frac{4}{3}ym^2)^{2-\epsilon/2}}$$

$$= \frac{1}{4}\pi^{d/2} \frac{\Gamma(2-\epsilon/2)}{(\frac{3}{4})^{1-\epsilon/2}} (\pi)^{d/2} \frac{\Gamma(1-\epsilon)}{\Gamma(2-\epsilon/2)} \int_0^1 \int_0^1 dxdy \int_{-\infty}^{\infty} d^d\vec{q} \frac{1}{(q^2+m^2)} \frac{y^{-\epsilon/2} \quad \vec{q}\cdot(\frac{2}{3}xy\vec{q})}{\{[\frac{4}{3}xy - (\frac{2}{3}xy)^2]q^2 + \frac{4}{3}ym^2\}^{1-\epsilon}}$$

$$= \frac{1}{4} \frac{(\pi)^d\Gamma(1-\epsilon)}{(\frac{3}{4})^{1-\epsilon/2}} \int_{-\infty}^{\infty} d^d\vec{q} \frac{q^2}{(q^2+m^2)} \frac{(\frac{2}{3}xy)y^{-\epsilon/2}}{\Lambda^{1-\epsilon}(q^2 + \frac{4y}{3\Lambda}m^2)^{1-\epsilon}}$$

$$= \frac{1}{4} \frac{(\pi)^d\Gamma(1-\epsilon)}{(\frac{3}{4})^{1-\epsilon/2}} \frac{2}{3} \frac{(\pi)^{d/2}\Gamma(2+\epsilon/2)\Gamma(-\frac{3}{2}\epsilon)}{\Gamma(1-\epsilon)\Gamma(1+\epsilon/2)} (m^2)^{\frac{3}{2}\epsilon} \int_0^1 \int_0^1 dxdy \frac{y^{-\epsilon/2}xy}{(\Lambda)^{1-\epsilon}} \left(\frac{4y}{3\Lambda}\right)^{\frac{3}{2}\epsilon}. \tag{B66}$$

Here $\Lambda = \frac{4}{3}xy - (\frac{2}{3}xy)^2$. The integration over x and y in Eq. (B66) is obtained as

$$\int_0^1\int_0^1 dxdy(\tfrac{4}{3})^{3\epsilon/2}\frac{xy^{1+\epsilon}}{[\frac{4}{3}(xy)-\frac{4}{9}(xy)^2]^{1+\epsilon/2}} = (\tfrac{4}{3})^{3\epsilon/2}\int_0^1\int_0^1 dxdy\frac{xy^{1+\epsilon}}{(xy)^{1+\epsilon/2}(\frac{4}{3})^{1+\epsilon/2}}\frac{1}{(1-\frac{xy}{3})^{1+\epsilon/2}}$$
$$= -3\int_0^1 dy\left[\frac{1-\frac{y}{3}}{y}\right] = -3I_E + O(\epsilon). \quad \text{(B67)}$$

Finally, we summarize the results of two three-loop calculations. One should observe that there are not any subdivergences in three-loop diagrams and therefore no such term as $\ln[(cm^2a^2)]^2$ exists. In the following summation of both three-loop calculation results, we demonstrate this observation by explicit calculations. By collecting all previous results, the contributions of FD $3l1$ and FD $3l2$ in ϵ^{-2} are given by

$$\text{FD } 3l1: -\frac{1}{2}\left[\frac{1}{4}\Gamma\left(\frac{-3\epsilon}{2}\right)\left(-\frac{8}{\epsilon}\right)\ln\left(\frac{4}{3}\right) - 2\frac{1}{8}\Gamma\left(\frac{-3\epsilon}{2}\right)\left(-\frac{4}{\epsilon}\right)\ln\left(\frac{4}{3}\right) - 2\frac{1}{8}\Gamma\left(\frac{-3\epsilon}{2}\right)4\ln\left(\frac{4}{3}\right)\frac{-1}{\epsilon}\right] = 0\,. \quad \text{(B68)}$$

$$\text{FD } 3l2: \quad -\frac{1}{4}\Gamma\left(\frac{-3\epsilon}{2}\right)4\ln\left(\frac{4}{3}\right)\left(-\frac{1}{\epsilon}\right) + 2\left(\frac{1}{8}\right)\Gamma\left(\frac{-3\epsilon}{2}\right)4\ln\left(\frac{4}{3}\right)\left(-\frac{1}{\epsilon}\right) = 0\,. \quad \text{(B69)}$$

These confirm our observation mentioned above. The consequent results of the leading divergences which contribute to Z_g are used in Eq. (22).

APPENDIX C: BASIC INTEGRALS

We define $\Xi(x,s) = \sum_{n=1}^{\infty}\frac{x^n}{n^s}$, and $\Phi(x,s) = \sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^s}$.
I_A: $\int_0^1 du\frac{\ln(1-\frac{u}{4})}{(1-\frac{u}{4})} = -2\ln^2(\frac{3}{4})$,
I_B: $\int_0^1 du\frac{\ln(1-\frac{u}{3})}{(1-\frac{u}{4})} = 2\ln^2(\frac{4}{3}) - 4[\Xi(\frac{1}{3},2)-\Xi(\frac{1}{4},2)]$,
I_C: $\int_0^1 du\frac{\ln u}{(1-\frac{u}{4})} = -4\Xi(\frac{1}{4},2)$,
I_D: $\int_0^1 du\frac{\ln(1-\frac{u}{4})}{u} = -\Xi(\frac{1}{4},2)$,
I_E: $\int_0^1 du\frac{\ln(1-\frac{u}{3})}{u} = -\Xi(\frac{1}{3},2)$,
I_F: $\int_0^1\int_0^1 dudv\frac{\ln(1-\frac{uv}{4})}{u} = -[\Xi(\frac{1}{4},2)+3\ln(\frac{4}{3})-1]$,
I_G: $\int_0^1 du\ln(u)\ln(1-\frac{u}{4}) = 2-3\ln(\frac{4}{3})-4\Xi(\frac{1}{4},2)$,
I_H: $\int_0^1\int_0^1 dudv\frac{\ln(1-\frac{u}{4-v})}{u} = 4\ln(\frac{4}{3})-\frac{5}{2}\ln(\frac{3}{2})+\frac{1}{2}$,
I_J: $\int_0^1 du\ln(1-\frac{u}{4}) = 3\ln(\frac{4}{3})-1$.

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